The General Linear Model

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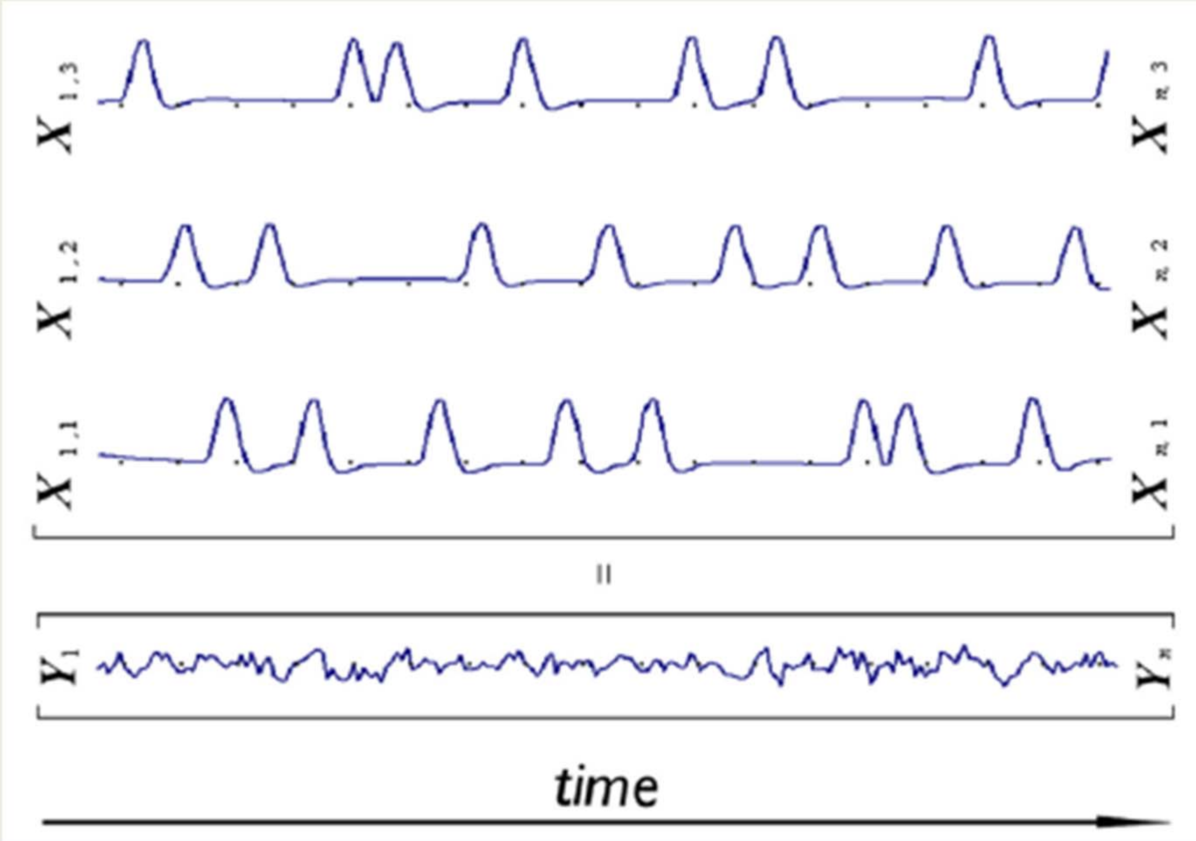
fMRI General Linear Model

Predicted time course for event type 1

Predicted time course for event type 2

Predicted time course for event type 3

BOLD time course in one voxel



measured time series

parameter estimates

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

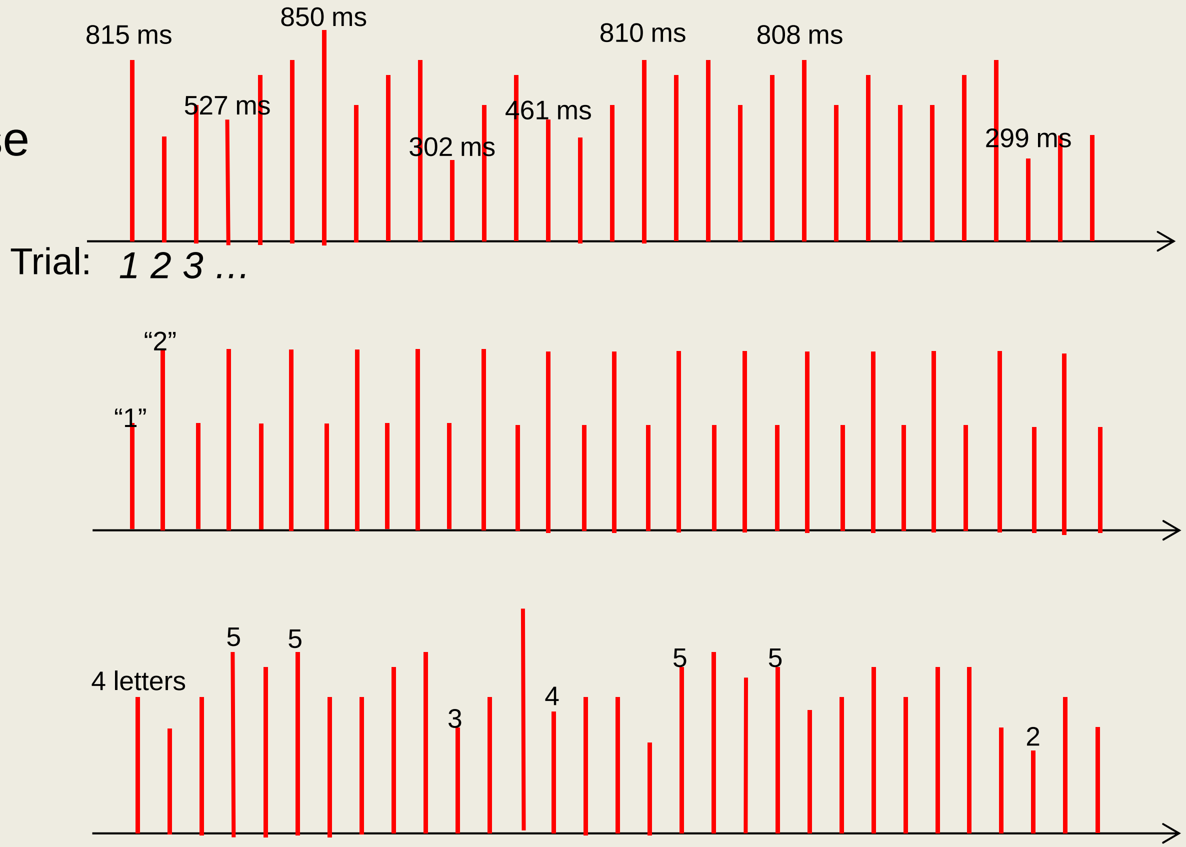
design matrix

Regression of Reaction Times

Which variables can explain these data?

Repetition?

Word length?



RTs

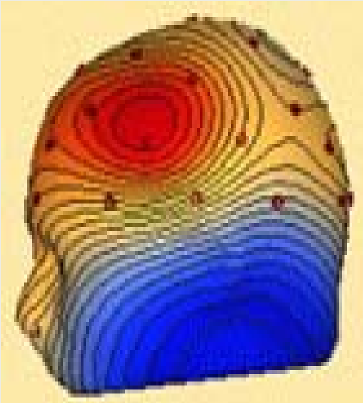
$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

regression coefficients

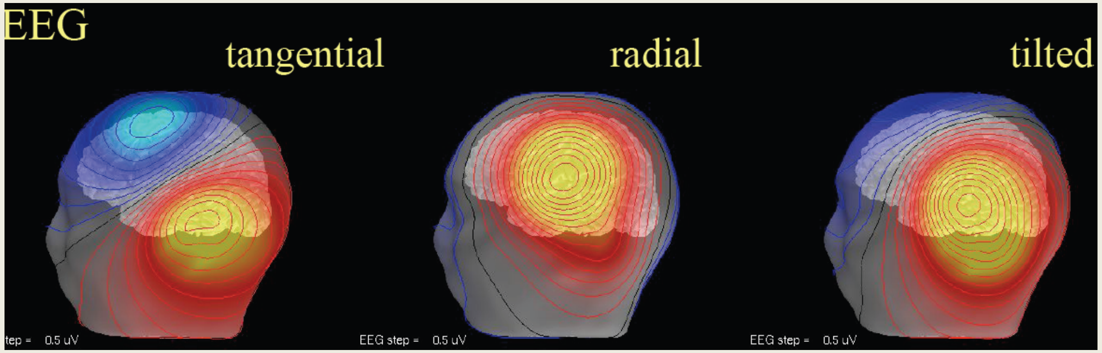
predictor variables

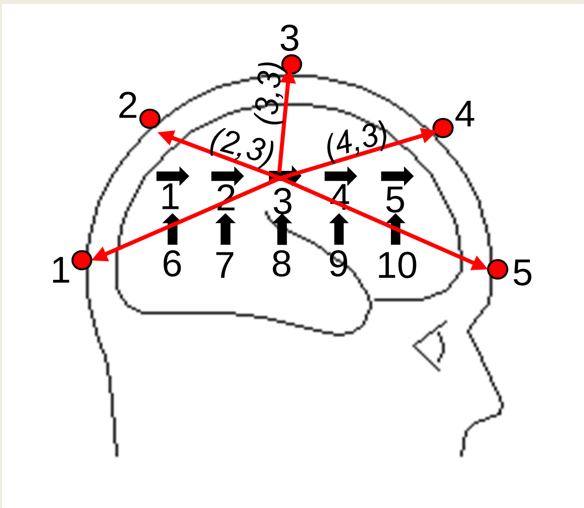
EEG/MEG “Inverse Problem”

Which sources explain this topography?



Maybe these?





data vector

source vector

$$\mathbf{d} = \mathbf{L}\mathbf{s}$$

Leadfield matrix

Basis Functions

Every vector with *n* elements can be decomposed into *n* different independent vectors

There are many such decompositions – some may be more useful than others

[1 2 3]
can be decomposed as

1*[1 0 0] + 2*[0 1 0] + 3*[0 0 1]

or

1*[1 1 1] + 1*[0 1 0] + 2*[0 0 1]

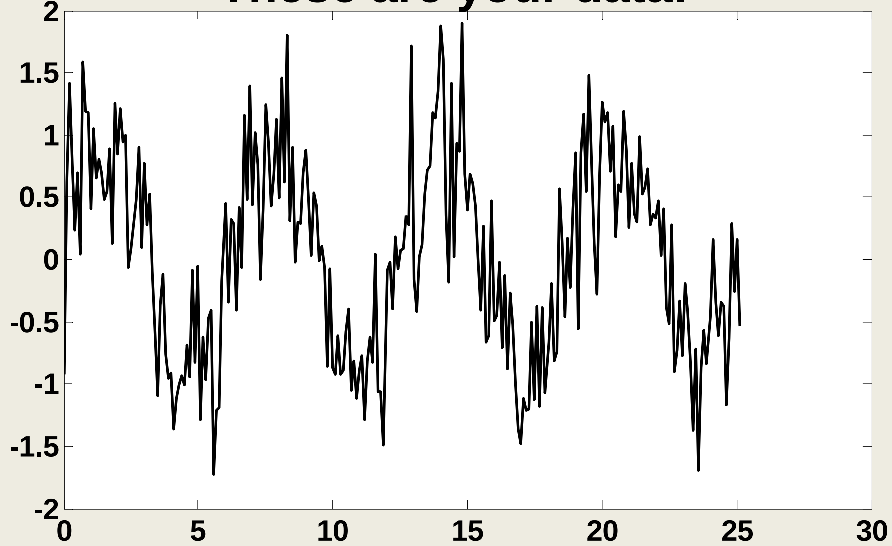
or

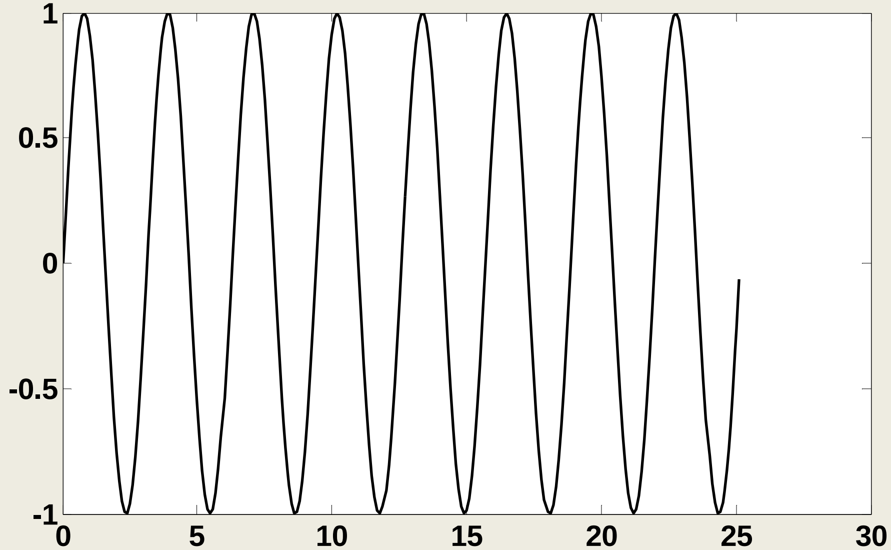
-1*[1 -1 0] + -1*[0 1 -1] + 2*[1 1 1]

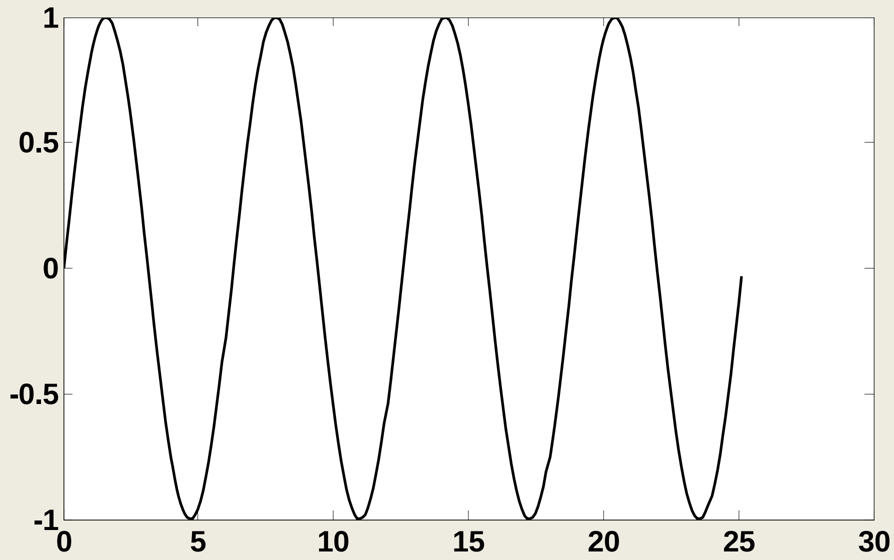
Example

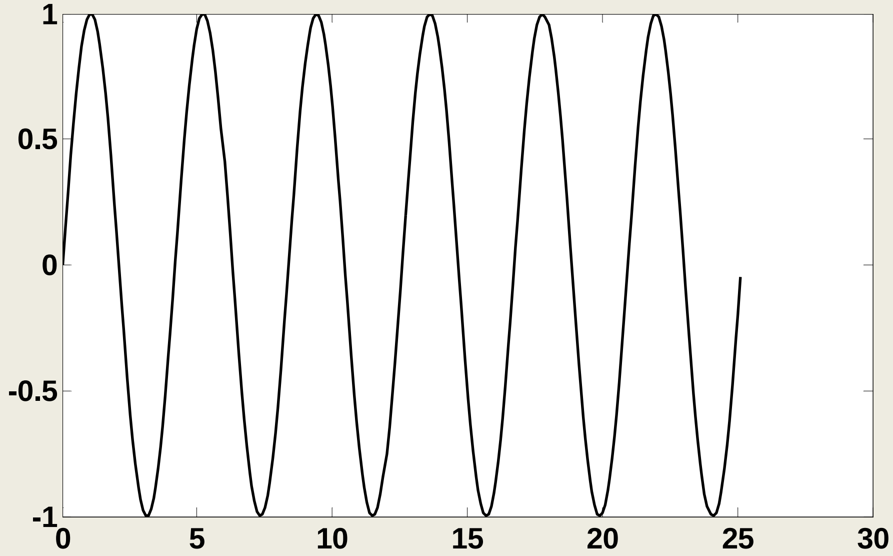
Choosing the right basis functions

These are your data:



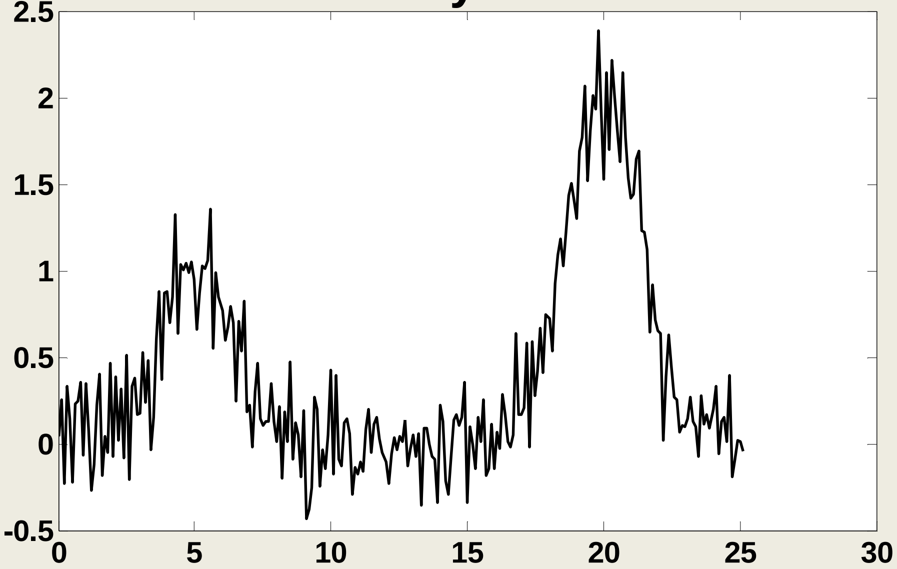


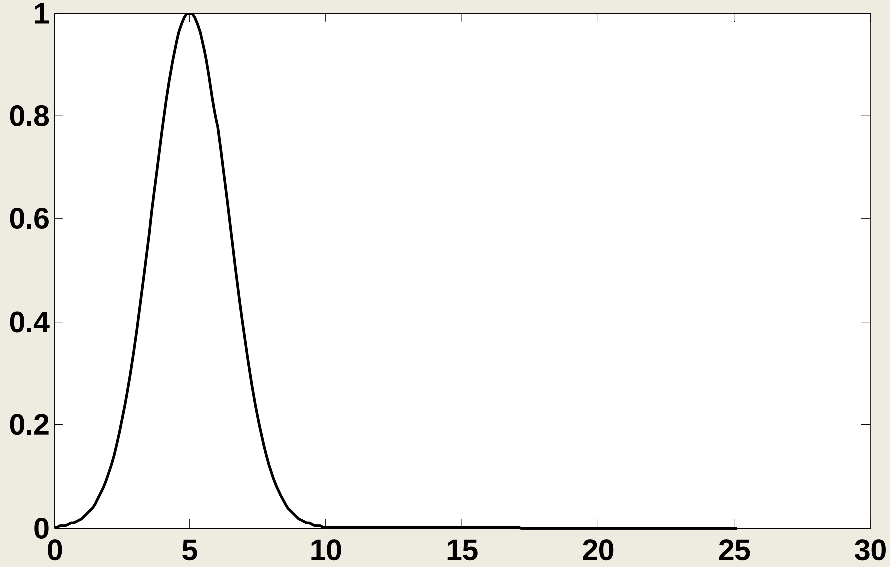


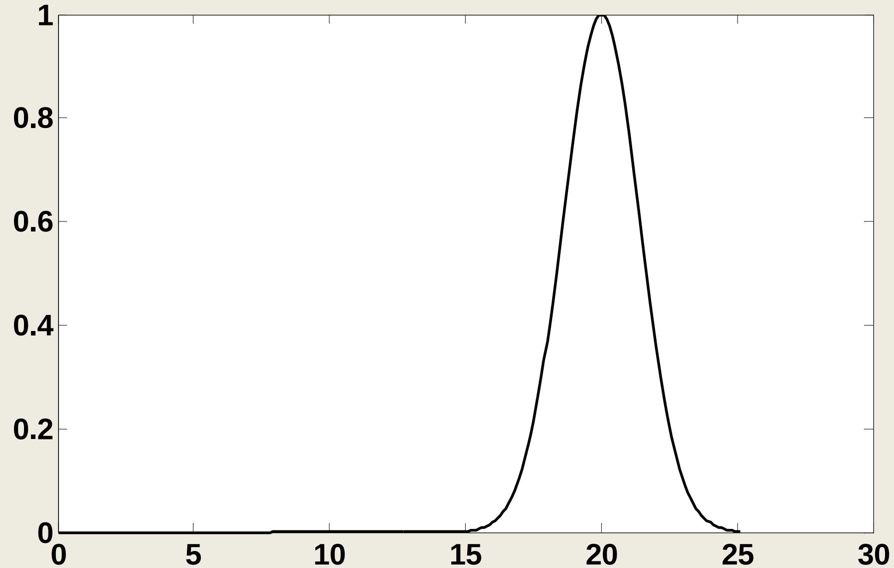


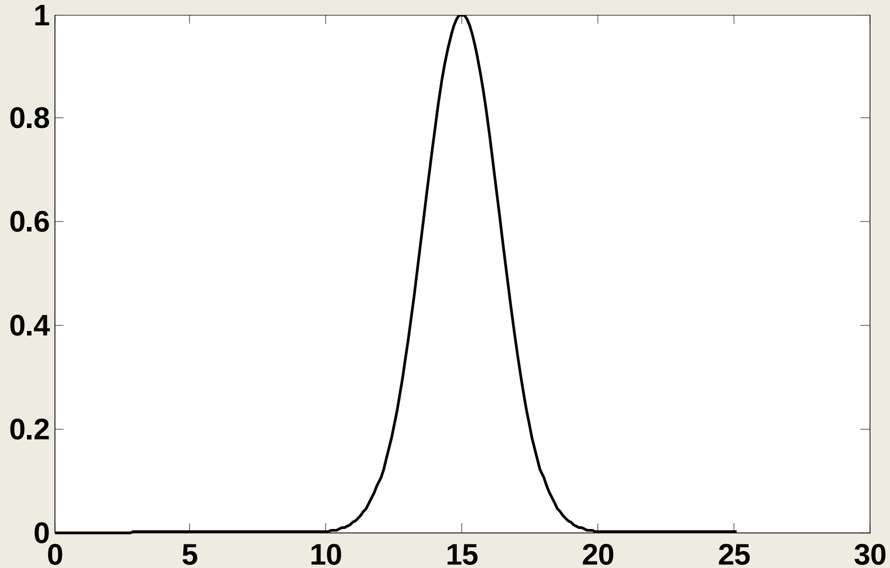
Choosing the right basis functions

These are your data:









The "Inverse Problem" – Parameter Estimation

Usually, we have

1) The data
2) A set of basis functions

=> We want to know:
How much do the different basis functions contribute to our measured data?

If possible, we want to describe the relationship between our desired parameters and the data in the framework of the General Linear Model:

$$\mathbf{d} = \mathbf{X}\mathbf{b}$$

For example:

1) Data: [1 2 3]
2) Basis functions: [1 -1 0] [0 1 -1] [1 1 1]
3) Problem: What are *a,b,c* for [1 2 3] = *a**[1 -1 0] + *b**[0 1 -1] + *c**[1 1 1]

Linear Equations

$$\mathbf{y} = \mathbf{x} + 2 \;\Rightarrow\; \mathbf{x} = ? \qquad \begin{pmatrix} y_1 \\ \ldots \\ y_j \\ \ldots \\ y_C \end{pmatrix} = \mathbf{y} = \mathbf{x} + 2 = \begin{pmatrix} x_1 + 2 \\ \ldots \\ x_j + 2 \\ \ldots \\ x_C + 2 \end{pmatrix} \;\Rightarrow\; \mathbf{x} = \mathbf{y} - 2 = \begin{pmatrix} x_1 - 2 \\ \ldots \\ x_j - 2 \\ \ldots \\ x_C - 2 \end{pmatrix}$$

$$\mathbf{y} = 2 * \mathbf{x} \;\Rightarrow\; \mathbf{x} = ? \qquad \begin{pmatrix} y_1 \\ \ldots \\ y_j \\ \ldots \\ y_C \end{pmatrix} = \mathbf{y} = 2 * \mathbf{x} = \begin{pmatrix} 2 * x_1 \\ \ldots \\ 2 * x_j \\ \ldots \\ 2 * x_C \end{pmatrix} \;\Rightarrow\; \mathbf{x} = \mathbf{y} / 2 = \begin{pmatrix} y_1 / 2 \\ \ldots \\ y_j / 2 \\ \ldots \\ y_C / 2 \end{pmatrix}$$

Linear Equations

Problem:

$$x_1 + x_2 = 1 \qquad x_1 = ?$$
$$x_1 - x_2 = 1 \qquad x_2 = ?$$

"Basis functions"

Matrix notation:

$$\begin{pmatrix} 1 * x_1 + 1 * x_2 \\ 1 * x_1 - 1 * x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{M} * \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution?

("2*x=1 => x = ½")

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\mathbf{M}} = \mathbf{M}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the "inverse" of a matrix?

Inverse Matrices

Definition: A matrix multiplied by its inverse is the identity matrix:

$$\mathbf{I} = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

(just like (1/3)*3 = 1)

$$\mathbf{M} * \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2*x=1

=> ½*2*x= ½

=> x = ½

$$\Downarrow$$

$$\underbrace{\mathbf{M}^{-1}\mathbf{M}}_{identity} * \mathbf{x} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} => \mathbf{x} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example

Linear Equations

Problem:

$$2x_1 + 0 * x_2 = 1 \quad x_1 = ?$$
$$0 * x_1 + 3x_2 = 1 \quad x_2 = ?$$

"Basis functions"

Matrix notation:

$$\begin{pmatrix} 2 * x_1 + 0 * x_2 \\ 0 * x_1 - 3 * x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{M} * \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\mathbf{M}} = \mathbf{M}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

Example

“Orthonormal” basis functions

Orthonormal: Orthogonal and of unit norm/length

For example:
[1 0] and [0 1] are orthonormal basis functions

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Downarrow$$

$$\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$\beta = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

No “inversion” necessary – just multiply basis functions to your data.

But: Often basis functions are not orthonormal

Problem of multiple linear regression:

If basis functions are correlated, the whole system of equations

needs to be taken into account

$\Rightarrow$ Matrix inversion is necessary

(“partialling out” variables)

“Linearly independent”: vectors are not perfectly correlated

“Orthogonal”: correction of vectors is exactly zero

What does “linear” actually mean?

An operation/system/function is “linear”

when if satisfies the following two conditions:

$$\text{Additivity}: f(x+y) = f(x) + f(y)$$

$$\text{Homogeneity}: f(\alpha x) = \alpha f(x)$$

Matrix multiplication is linear:

$$\mathbf{M} * (\alpha \mathbf{X} + \beta \mathbf{Y}) = \alpha \mathbf{M}\mathbf{X} + \beta \mathbf{M}\mathbf{Y}$$

But again…

$$\mathbf{XY} \text{ is often not } \mathbf{YX}!$$

Example

“Overdetermined Problem” (e.g. Regression)

$$1*x_1 + 1*x_2 = 1$$
$$2*x_1 + 1*x_2 = -1$$
$$2*x_1 + 2*x_2 = 2$$
$$3*x_1 + 1*x_2 = 0$$
$$3*x_1 + 2*x_2 = 1.5$$
$$3*x_1 + 3*x_2 = 2.5$$

$$x_1 = ?$$
$$x_2 = ?$$

$$\begin{pmatrix}1\\2\\2\\3\\3\\3\end{pmatrix} * x_1 + \begin{pmatrix}1\\1\\2\\1\\2\\3\end{pmatrix} * x_2 = \underbrace{\begin{pmatrix}1 & 1\\2 & 1\\2 & 2\\3 & 1\\3 & 2\\3 & 3\end{pmatrix}}_{\mathbf{M}} \begin{pmatrix}x_1\\x_2\end{pmatrix} = \mathbf{Mx} = \underbrace{\begin{pmatrix}1\\-1\\2\\0\\1.5\\2.5\end{pmatrix}}_{\mathbf{d}} = \mathbf{d}$$

M is not invertible, there is no unique solution for **x**.

We can find the **x** that minimises the least-squares error: $\left\|\mathbf{Mx} - \mathbf{d}\right\|^2 = \min$

The matrix that provides this least-squares solution is the “pseudoinverse” of **M:** $\mathbf{M}^-$
(in Matlab: “pinv”)

$$\mathbf{M}^- = \begin{matrix} -0.0526 & 0.1579 & -0.1053 & 0.3684 & 0.1053 & -0.1579 \\ 0.1158 & -0.1474 & 0.2316 & -0.4105 & -0.0316 & 0.3474 \end{matrix}$$

"Underdetermined Problem" (e.g. EEG/MEG Inverse Problem)

$$1*x_1 + 1*x_2 + 1*x_3 = 1$$
$$1*x_1 + 2*x_2 + 3*x_3 = -1$$

$$x_1 = ?$$
$$x_2 = ?$$
$$x_3 = ?$$

$$\begin{pmatrix}1\\1\end{pmatrix}*x_1 + \begin{pmatrix}1\\2\end{pmatrix}*x_2 + \begin{pmatrix}1\\3\end{pmatrix}*x_3 = \begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \mathbf{Mx} = \begin{pmatrix}1\\-1\end{pmatrix} = \mathbf{d}$$

M is not invertible, there is no unique solution for **x**.
We can find the **x** that minimises the "norm" of the solution:

$$\mathbf{Mx} = \mathbf{d}$$
$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2$$

The matrix that provides this least-squares solution is the "pseudoinverse" of **M:** $\mathbf{M}^-$
(in Matlab: "pinv")

$$\mathbf{M}^- = \begin{matrix}1.3333 & -0.5000\\0.3333 & 0.0000\\-0.6667 & 0.5000\end{matrix}$$