Functions and Calculus

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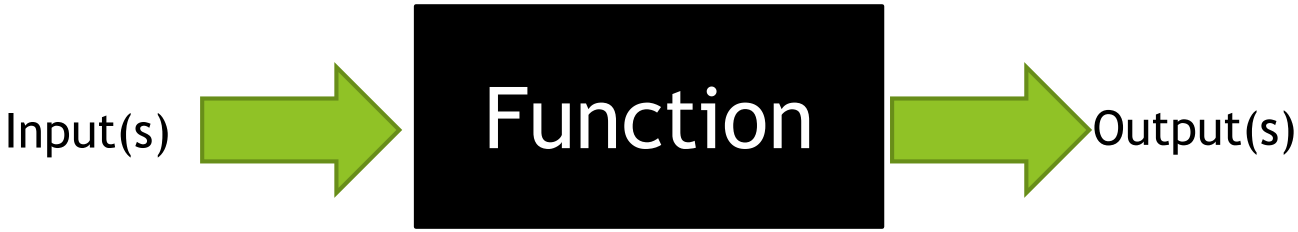
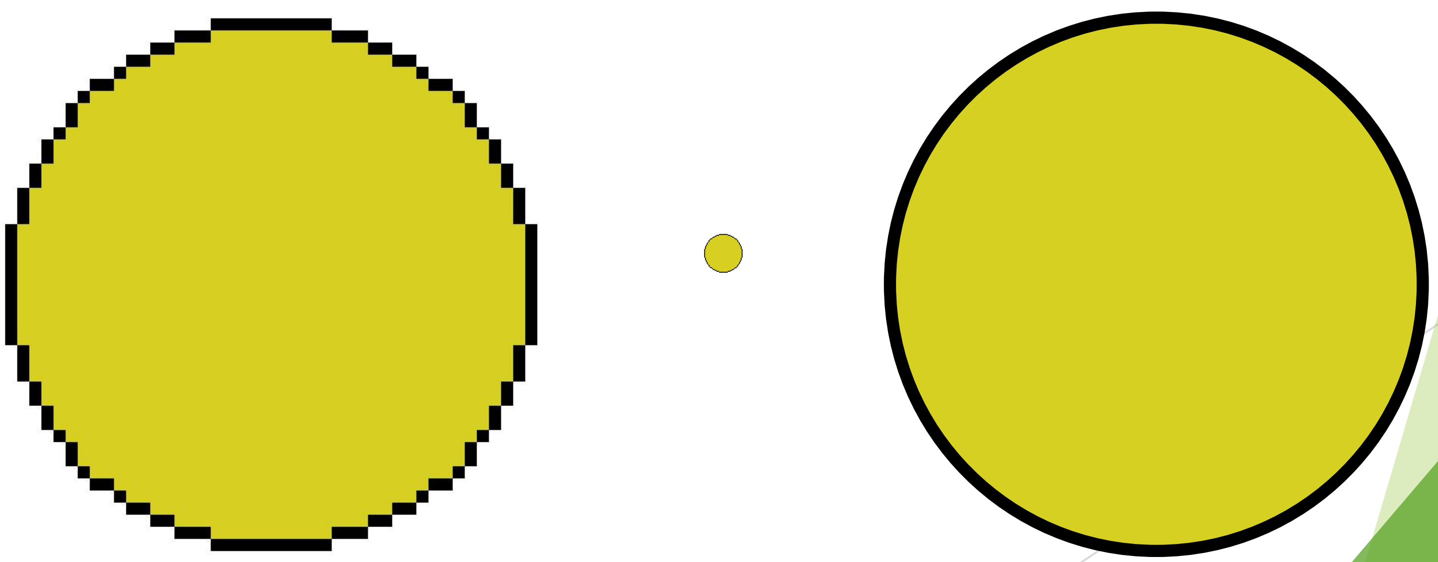
What is a function?



- Takes some input, and gives an output
- Only 1 rule: for any input, there can only be a maximum of one output. There cannot be two outputs for one input
- Otherwise, anything else goes:
  - Multiple inputs can give the same output
    - $f(x) = x^2\,,\ \ f(2) = 4 = f(-2)$
  - Some functions don’t give all outputs
    - $f(x) = \sin(x)\,,\ \ -1 \leq f(x) \leq 1$
  - Some inputs don’t give any output
    - $f(x) = \sqrt{x},\ \ f(-1) = undefined^*$

Why do we need functions?

- Figure out the underlying pattern
- Whenever we collect data, we are collecting discrete values for discrete inputs ➔ But reality is continuous
- Ease of communication
- Easier/Cleaner to manipulate



Plotting Functions

- For 1D functions:
  - x = -3:0.1:3; %(define your x steps)
  - y = x.*x; %(define y)
  - plot(x,y)
- For 2D functions:
  - x1=linspace(-10,10,1001); %(define your x steps)
  - y1=linspace(-10,10,1001)'; %(define your y steps)
  - [X,Y]=meshgrid(x1,y1); %(combine both into a matrix)
  - z = X.^2+Y.^2;
  - imagesc(x1,y1,z) % (2D plot)
  - surf(x1,y1,z); shading interp; colorbar % (3D plot)

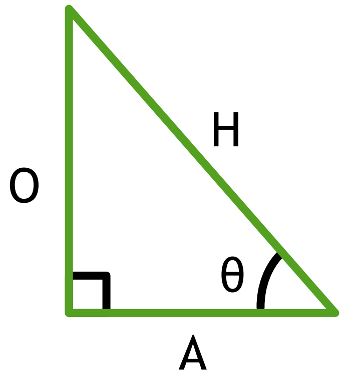
Popular Functions

- Easy to manipulate (differentiate/integrate/combine)
- Strongly representative of reality

Popular Functions (Polynomials)

- Extremely easy to manipulate
- Useful for regression, power laws and fitting unknown models
  - E.g. $f(x) = 2 * x$
  - E.g. $f(x) = x^4 + 3$
- Works with negative or fractional exponents
  - E.g. $f(x) = x^{-3}$
  - E.g. $f(x) = x^{1/4}$

Popular Functions (Trigonometric)



- $\sin(\theta) = \frac{O}{H}$
- $\cos(\theta) = \frac{A}{H}$
- $\tan(\theta) = \frac{O}{A} = \frac{\sin(\theta)}{\cos(\theta)}$

- Any complex function can be decomposed into a sum of trigonometric functions (Fourier transform)
- Easy to manipulate
- Important for oscillations, waves and filters

Popular Functions (Wave Eqns)

$$f(x,t) = A * \cos(kx + \omega t + \varphi)$$

A: amplitude, k: wavenumber, ω: frequency, φ: phase

Try plotting $f(x) = A * \cos(kx + \varphi)$!

How about a wave in 2D?

$$f(x,y) = A * \cos\left(k_x x + k_y y + \varphi\right)$$

$$f(x,y) = A * \cos\left(k_x x^2 + k_y y^2 + \varphi\right)$$

```
x1=linspace(-10,10,1001); %(define your x steps)
y1=linspace(0,20,1001)'; %(define your y steps)
[X,Y]=meshgrid(x1,y1); %(combine both into a matrix)

wave1D = cos(x1);
plot(x1,wave1D);

wave2D1 = cos(X + 2*Y);
wave2D2 = cos(0.1*X.^2 + 0.1*Y.^2);
figure; imagesc(x1,y1,wave2D1) % (2D plot)
figure; imagesc(x1,y1,wave2D2) % (2D plot)
```

Popular Functions (Wave Eqns)

Try plotting interference patterns!

$$f(x,y) = A * \cos\left(k_x(x-\delta)^2 + k_y y^2\right) + A * \cos\left(k_x(x+\delta)^2 + k_y y^2\right)$$

```
x1=linspace(-10,10,1001); %(define your x steps)
y1=linspace(0,20,1001)'; %(define your y steps)
[X,Y]=meshgrid(x1,y1); %(combine both into a matrix)

wave1 = cos(0.1*(X-2).^2 + 0.1*Y.^2);
wave2 = cos(0.1*(X+2).^2 + 0.1*Y.^2);
interference = wave1+wave2;
figure
imagesc(x1,y1, wave1) % (wave 1)
figure
imagesc(x1,y1, wave2) % (wave 2)
figure
imagesc(x1,y1, interference) % (sum of the two)
```

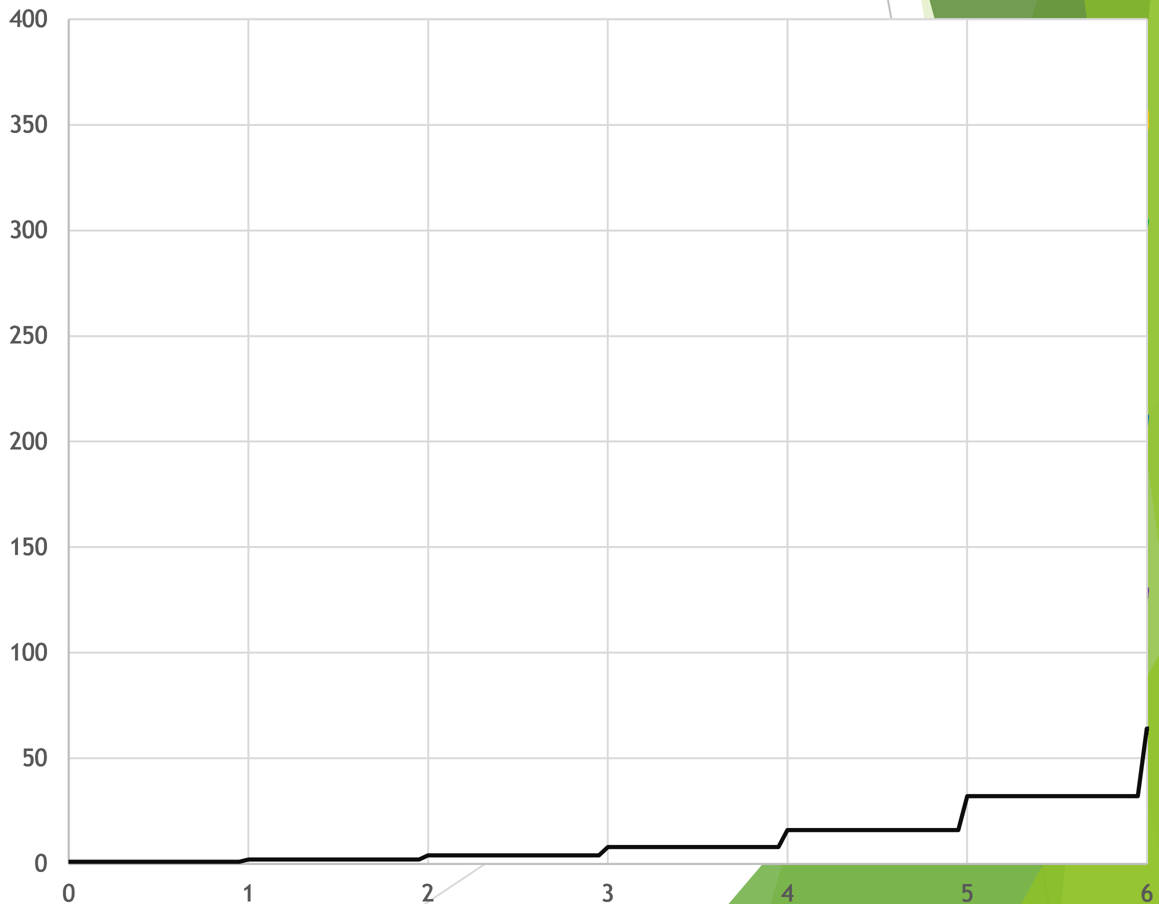
Popular Functions (Exponents)

- e = 2.718281828
- Extremely important for growth and decay processes, likelihoods and Bayesian estimations

You have £1. You can either receive:

a) 1x your current money every day
b) 0.5x your current money every 12 hours
c) 0.25x your current money every 6 hours

What is the maximum rate you can increase your money?

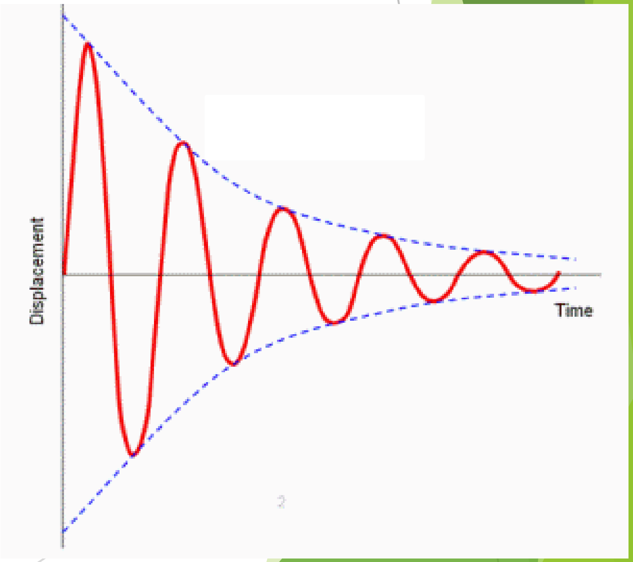


Combining Functions

- Functions are like toppings on a burger → you can have them in nearly any combination, its just that some combinations are better than others.
- E.g. Decaying oscillation of a pendulum
  - $f(t) = e^{-bt} * A * \cos(\omega t)$

    Decay function Oscillation function
  - What happens when $\omega > \mathrm{k}$ and $\omega < k$?

  - t = 0:0.1:6; %(define your time steps)
  - y1 = exp(-t).*cos(5*t); %($\omega > \mathrm{k}$)
  - y2 = exp(-t).*cos(0.2*t); %($\omega < \mathrm{k}$)
  - figure; hold on;
  - plot(t,y1,'r')
  - plot(t,y2,'k')

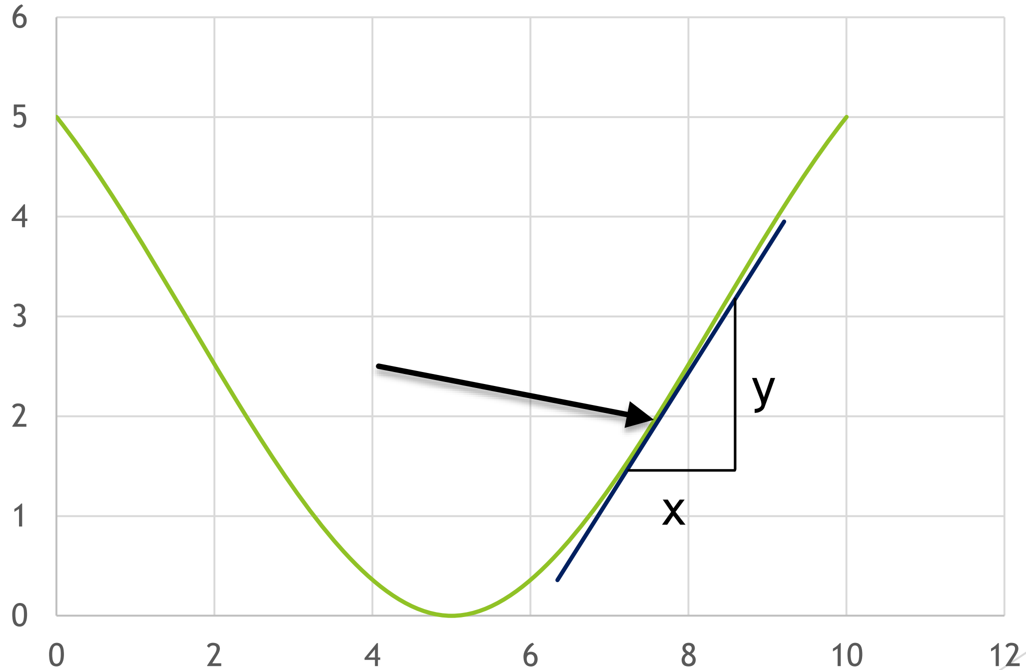


Why do we need calculus?

- Present all throughout our everyday life
  - Calculating the minimum energy state for a system
  - Optimizing a set of equations
  - Determining the rate of growth of a population
  - Calculating total area under a curve
- So that we don't do things like:
  - A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves (http://care.diabetesjournals.org/content/17/2/152.abstract)

Differentiation

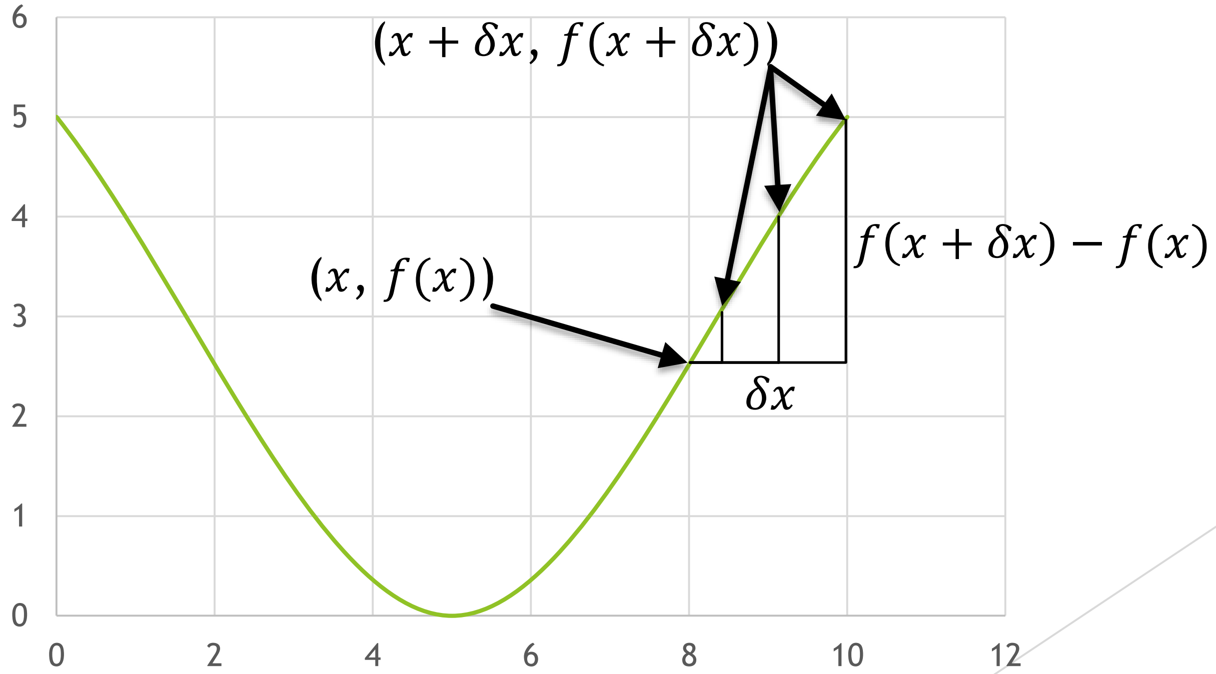
- The derivative describes the local rate of change of a function
- Notations: $f'(x), \dot{f}(x), \frac{df(x)}{dx}, \frac{\partial f(x,t)}{\partial x}$



$$f'(x) = \frac{y}{x}$$

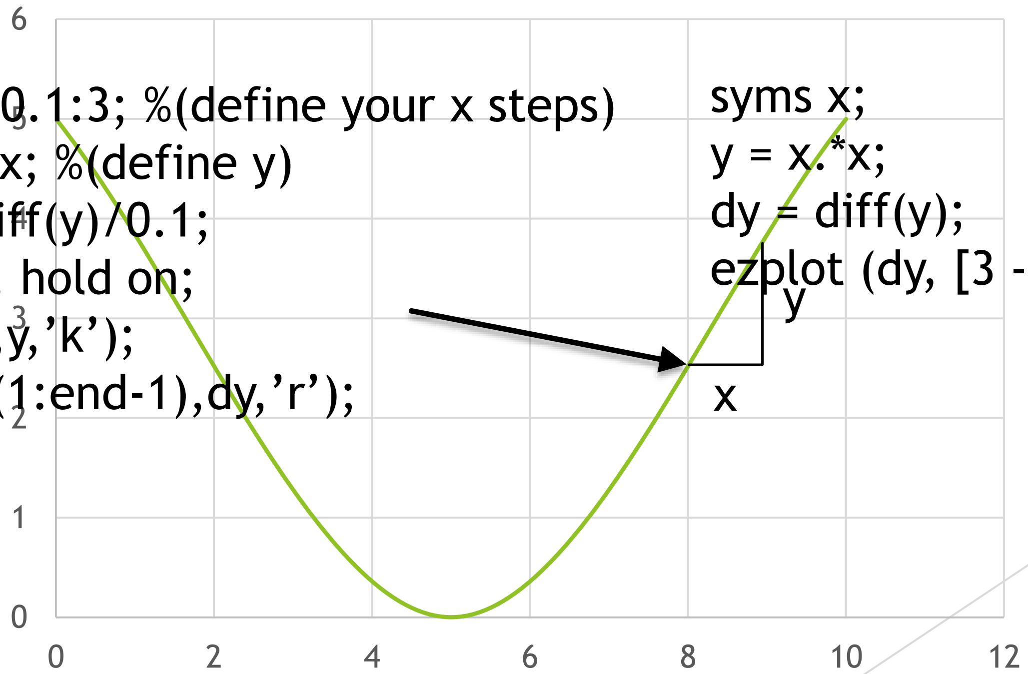
Differentiation (Mathematically Rigorous)

- $\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$



Differentiation

- In matlab:
  - diff(y) (approximation using step size of 1 cell)
  - Alternatively, you can use the symbolic toolbox instead
- Alternatively, if you know the function, you can calculate the derivative analytically
  - www.wolframalpha.com




```
x = -3:0.1:3; %(define your x steps)
y = x.*x; %(define y)
dy = diff(y)/0.1;
figure; hold on;
plot(x,y,'k');
plot(x(1:end-1),dy,'r');
```

```
syms x;
y = x.*x;
dy = diff(y);
ezplot (dy, [3 -3]);
```

$$\mathrm{diff}(f(x)) = \frac{y}{x}$$

Some common derivatives

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^2 + 2) = 2x$
- $\frac{d}{dx}(x^a) = a * x^{a-1}$
- $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- $\frac{d}{dx}(\sin(ax)) = a * \cos(ax)$
- $\frac{d}{dx}(e^x) = e^x$

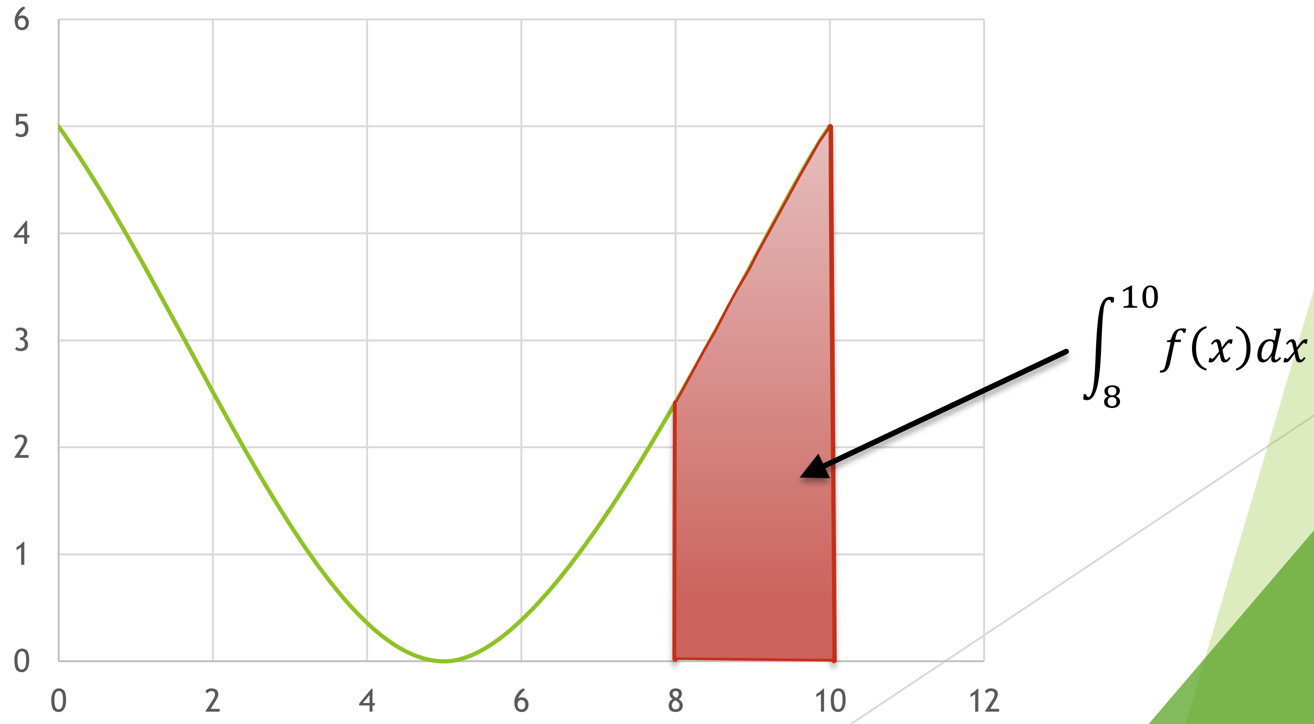
- $\int 1\, dx = x + c$
- $\int x\, dx = \frac{1}{2}x^2 + c$
- $\int x^a\, dx = \frac{1}{a+1}x^{a+1} + c$
- $\int \sin(x)\, dx = -\cos(x) + c$
- $\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$
- $\int e^x\, dx = e^x$

Integration (Indefinite Integral)

- $F(x) = \int f(x)dx$ (indefinite integral)
  - Mathematical concept as the opposite of differentation
  - If $g(x) = \int f(x)dx$, then $g'(x) = f(x)$
  - If $g'(x) = f(x)$ then $\int f(x)dx = g(x) + c$,

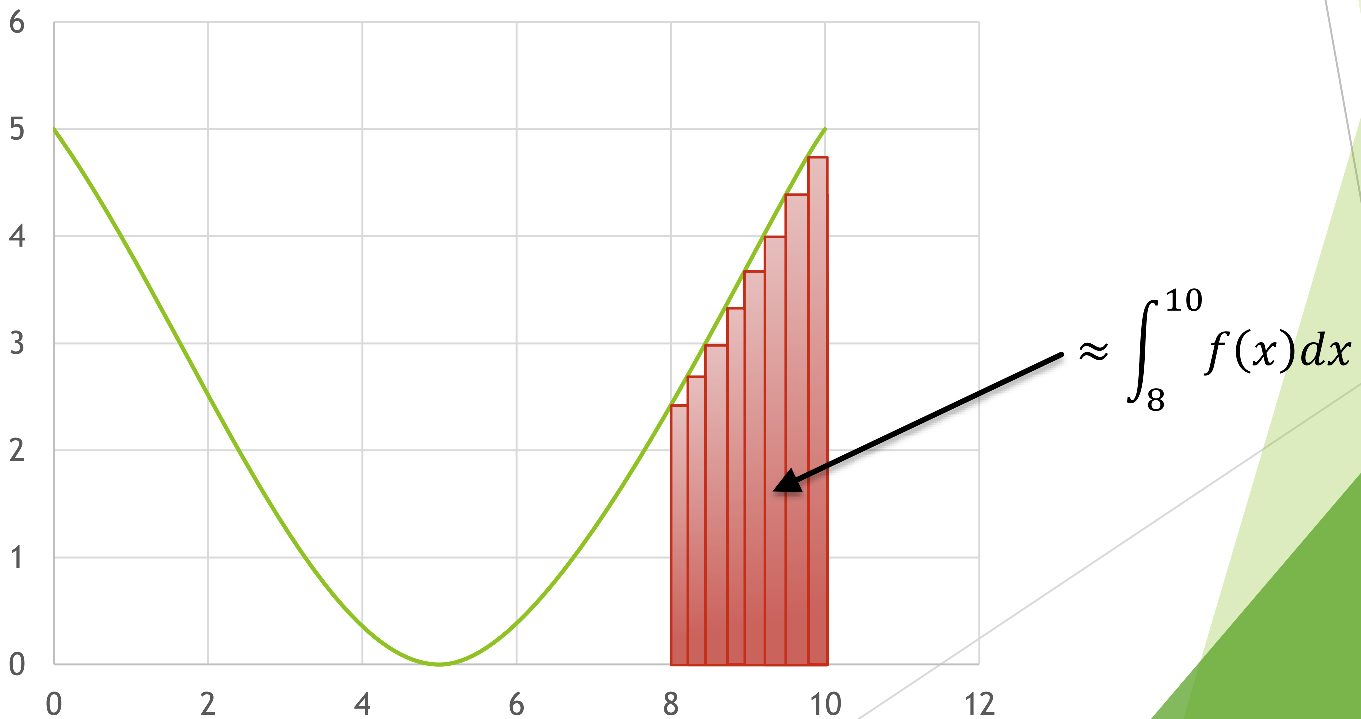
Integration (Definite Integral)

- $\int_a^b f(x)dx$
  - $\int_a^b f(x)dx = F(b) - F(a)$ (where $F(x) = \int f(x)dx$)
  - Gives the area under a graph from a to b



Integration (Definite Integral) (nearly mathematically rigorous)

- $\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum_{n=0}^{\frac{b-a}{\delta x}} (f(a + n * \delta x) * \delta x)$



Symbolic Toolbox

- syms x; %(Defines x to be a symbol instead of a variable)
  - fun= exp(-x).*cos(x); %(fun is now a function since x is a symbol)
- We can now differentiate/intergrate the function as per normal
  - dfun = diff(fun);
  - Q = int(fun, x, [xmin xmax]);
  - double(Q); %(convert Q into a numerical answer)

Integration (Example)

```
syms n; %(Defines x to be a symbol instead of a variable)
fun= n;
count= 1;
for i=0:0.1:3
        integral(count) = int(fun, n,[0 i]);
        count  = count + 1;
end
x = 0:0.1:3;
y = x;
figure; hold on;
plot(x,y,’k’);
plot(x,integral,’r’);

Int_y = 0.5*x.*x;
plot(x,Int_y);
```

Integration (Example)

```
syms n; %(Defines x to be a symbol instead of a variable)
fun= cos(n);
count= 1;
for i=0:0.1:3
        integral(count) = int(fun, n,[0 i]);
        count  = count + 1;
end
x = 0:0.1:3;
y = cos(x);
figure; hold on;
plot(x,y,'k');
plot(x,integral,'r');

Int_y = sin(x);
plot(x,Int_y);
```

Integration (Example)

- Try finding $\int_{-\pi/2}^{\pi/2} \tan(x)\, dx$
  - y= tan(x);
  - int(y, x, -pi/2, pi/2);
- If we simply plot it out, the answer is obvious!
  - figure
  - hold on
  - ezplot(y,[-pi/2 pi/2]);
  - ezplot(0*x,[-pi/2 pi/2]); %(plot x axis)