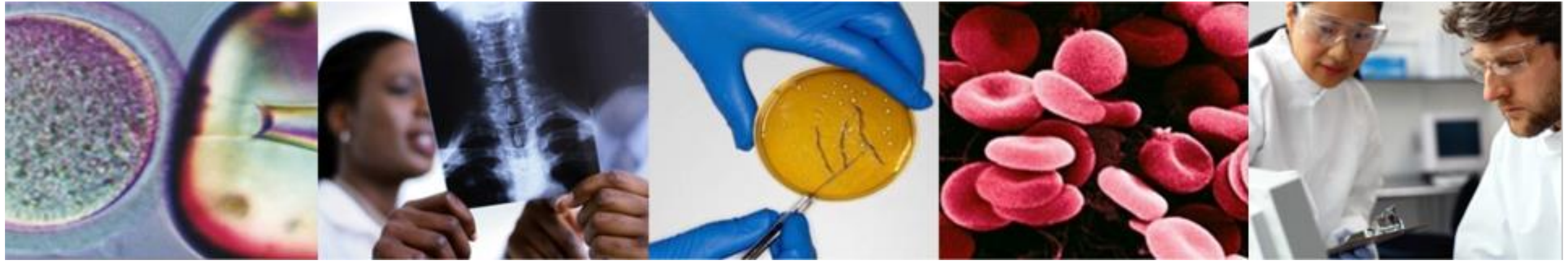
MRC Cognition and Brain Sciences Unit
75th ANNIVERSARY 1944 - 2019
UNIVERSITY OF CAMBRIDGE



The General Linear Model I

Olaf Hauk

olaf.hauk@mrc-cbu.cam.ac.uk

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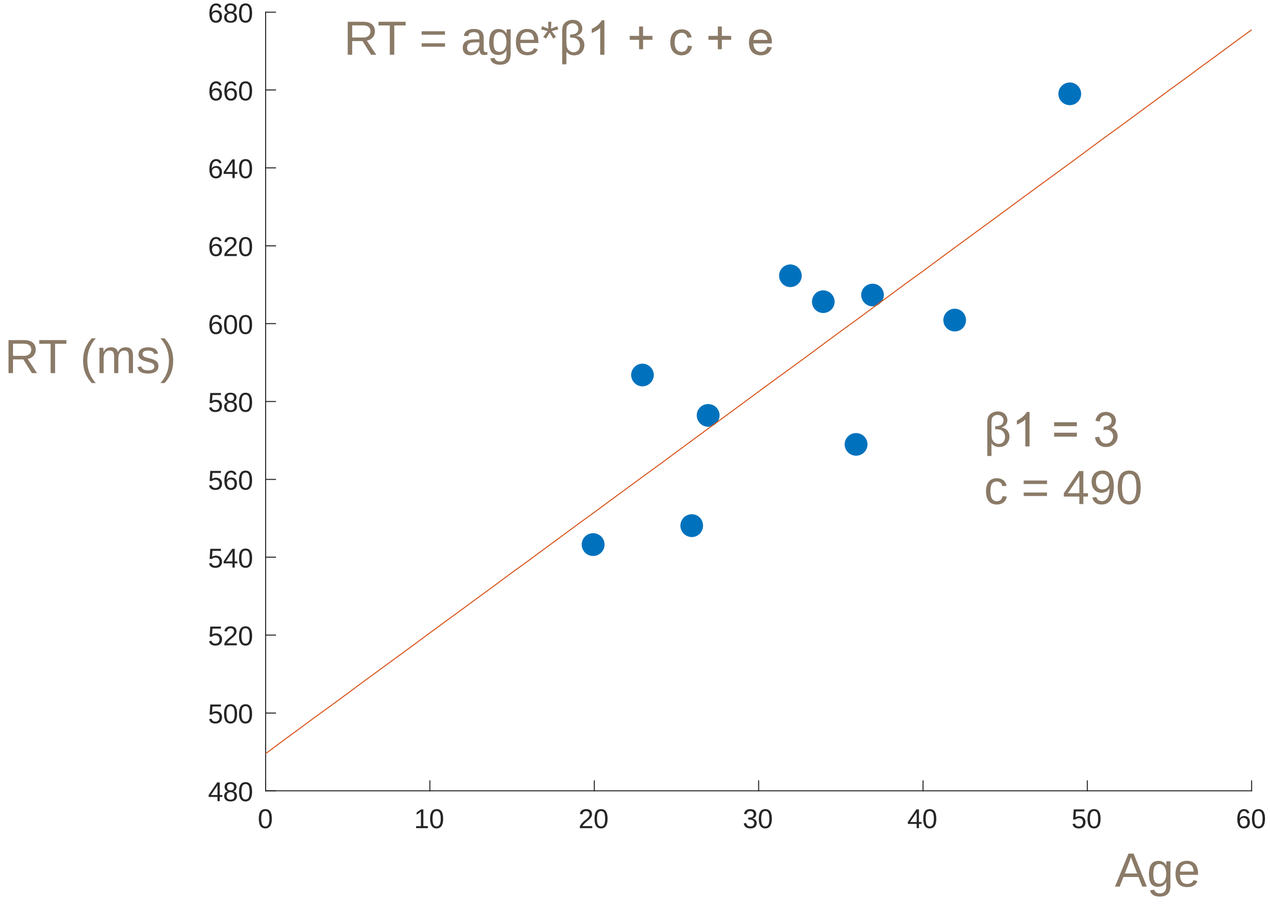
Linear Regression



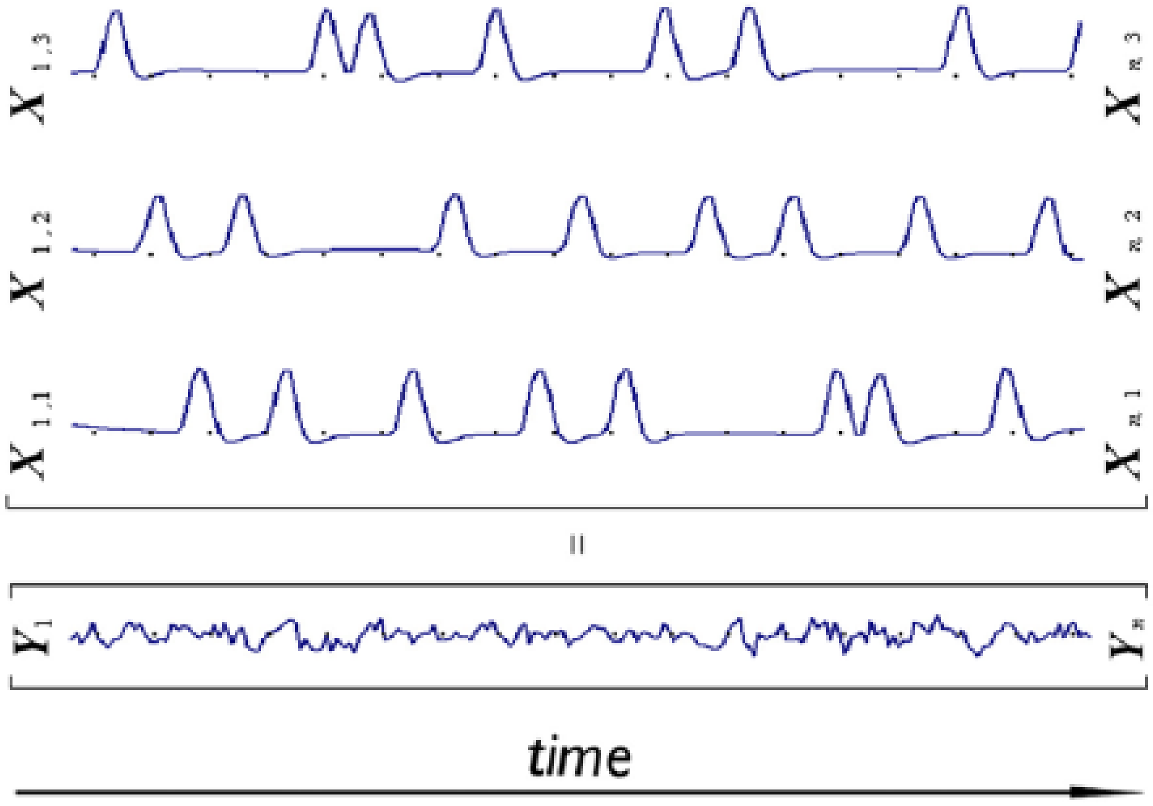
fMRI General Linear Model

Predicted time course for event type 1

Predicted time course for event type 2

Predicted time course for event type 3

BOLD time course in one voxel



measured time series

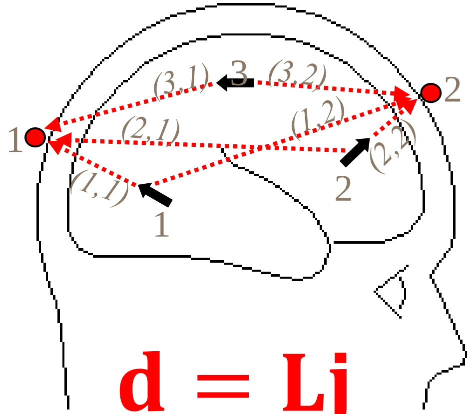
$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

parameter estimates

design matrix

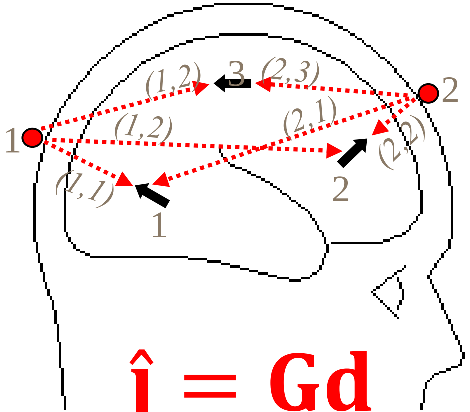
The EEG/MEG Inverse Problem Can Be Linear

Forward Problem



$$\mathbf{d} = \mathbf{L}\mathbf{j}$$

Linear Inverse Problem



$$\hat{\mathbf{j}} = \mathbf{G}\mathbf{d}$$

data "leadfield" dipoles

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0.3 \\ 0 & 1 & -0.3 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}$$

? inversion ⇒

dipoles inverse data

$$\begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \begin{pmatrix} 1.5034 & 0.1241 \\ 0.2483 & 0.9379 \\ 0.8276 & -0.2069 \end{pmatrix} * \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Let's start with a simple problem

$$y = x * \beta$$
$$\Rightarrow$$
$$\beta = y / x$$

(y, x, β: scalar numbers)

$$9 = 3 * \beta$$
$$\Rightarrow$$
$$\beta = (1/3) * 9 = 3$$

This is the simplest form of the GLM

$$\mathbf{y} = \mathbf{X} * \boldsymbol{\beta}$$

We are looking for an "estimator"/"operator"

$$\boldsymbol{\beta} = \mathbf{G} * \mathbf{y}$$

Note:
I write scalar numbers as non-bold italics, e.g. *x*, vectors as bold small letters, e.g. **x**, and matrices as bold capital letters, e.g. **X**.

There may be more than one

$$y_1 = x_1 * \beta_1$$
$$y_2 = x_2 * \beta_2$$

This can be written as

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \mathbf{X} * \boldsymbol{\beta}$$

Solution:

$$\boldsymbol{\beta} = \begin{pmatrix} 1/x_1 & 0 \\ 0 & 1/x_2 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{X}^{-1} * \mathbf{y}$$

Interpretation in terms of “basis functions”

$$y_1 = x_1 * \beta_1$$
$$y_2 = x_2 * \beta_2$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \mathbf{X} * \boldsymbol{\beta}$$

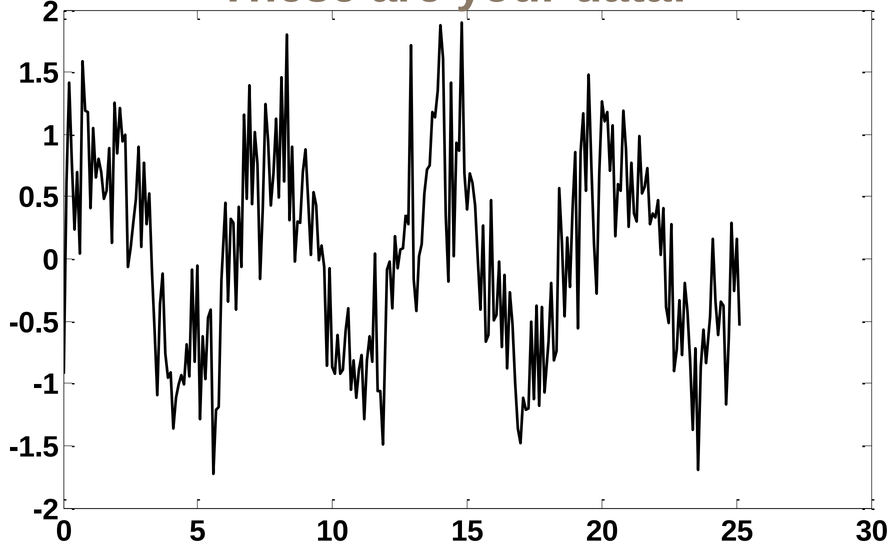
can be interpreted as

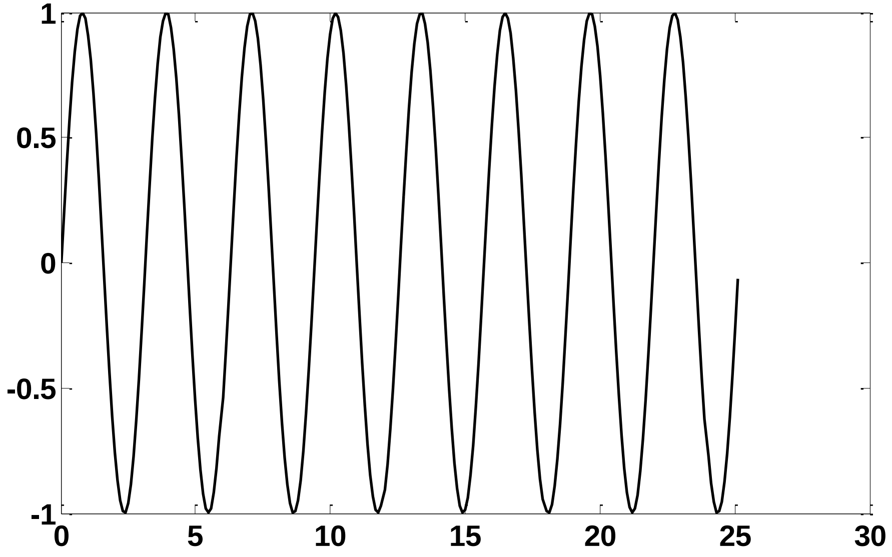
$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} * \beta_1 + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} * \beta_2$$

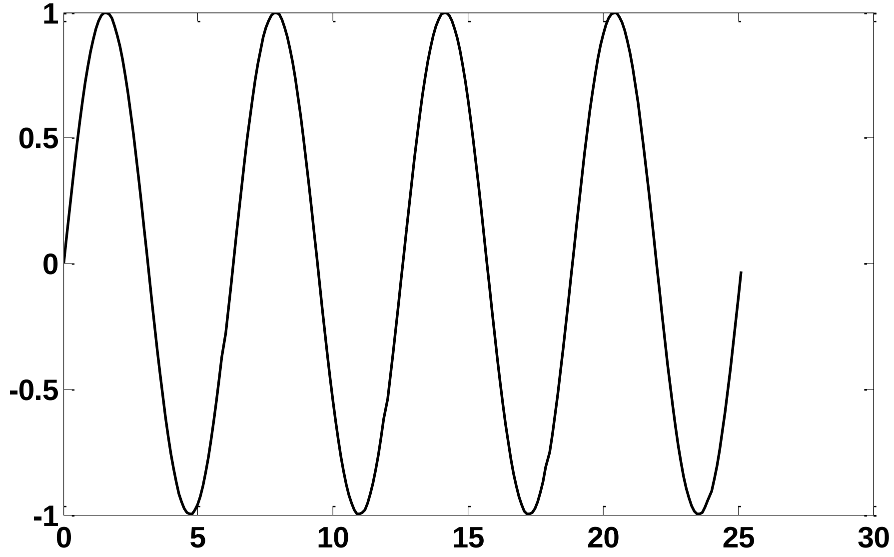
Orthogonal “basis functions”

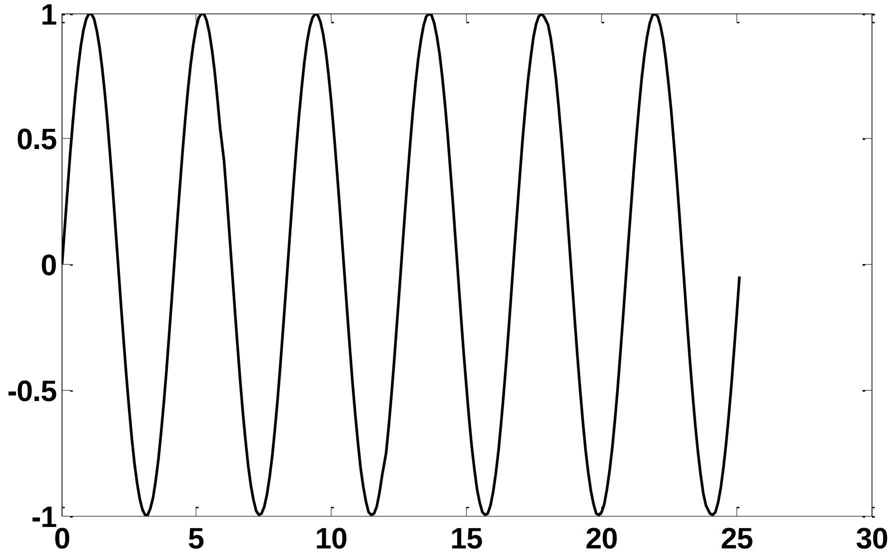
Choosing the right basis functions

These are your data:









Choosing the right basis functions

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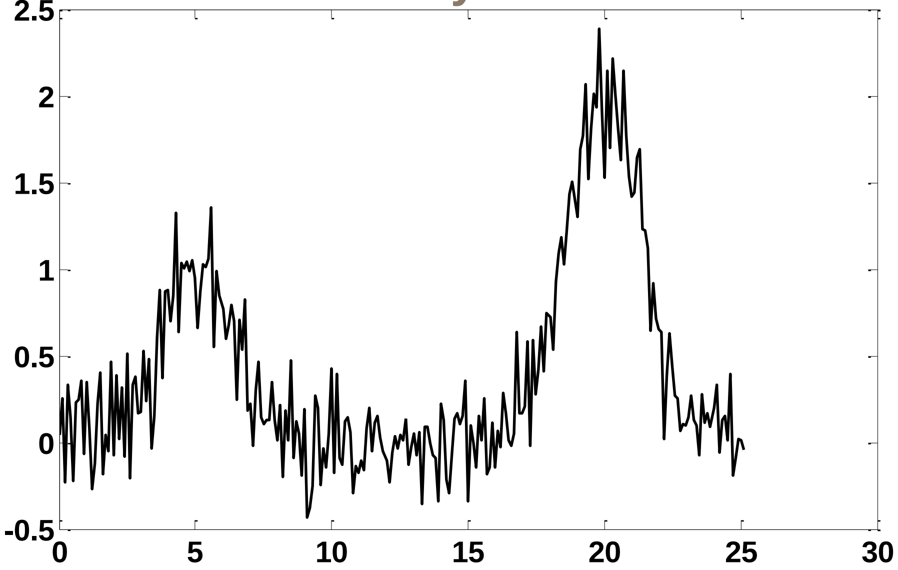
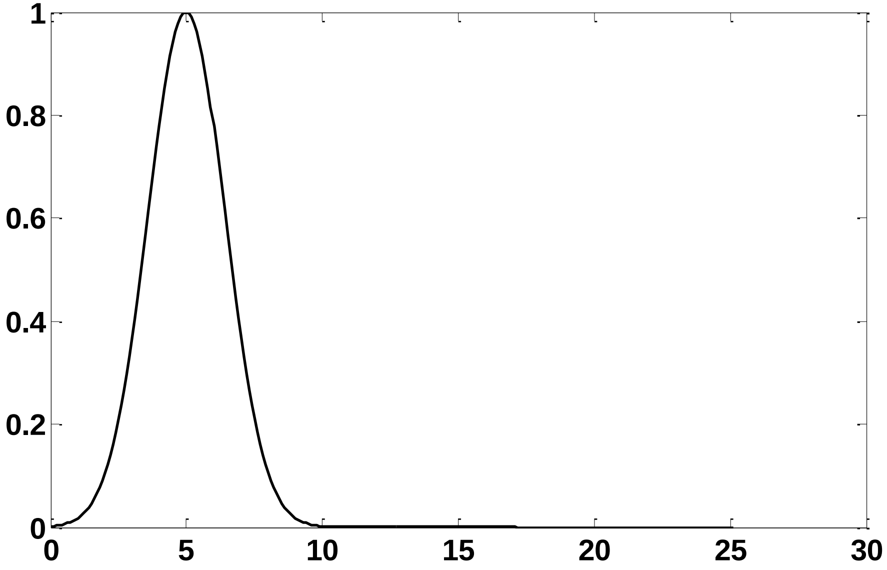
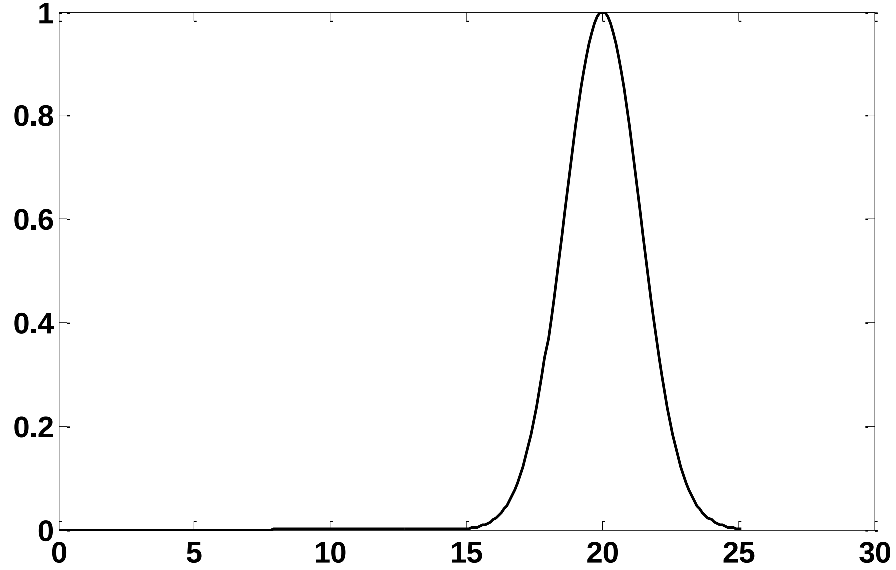
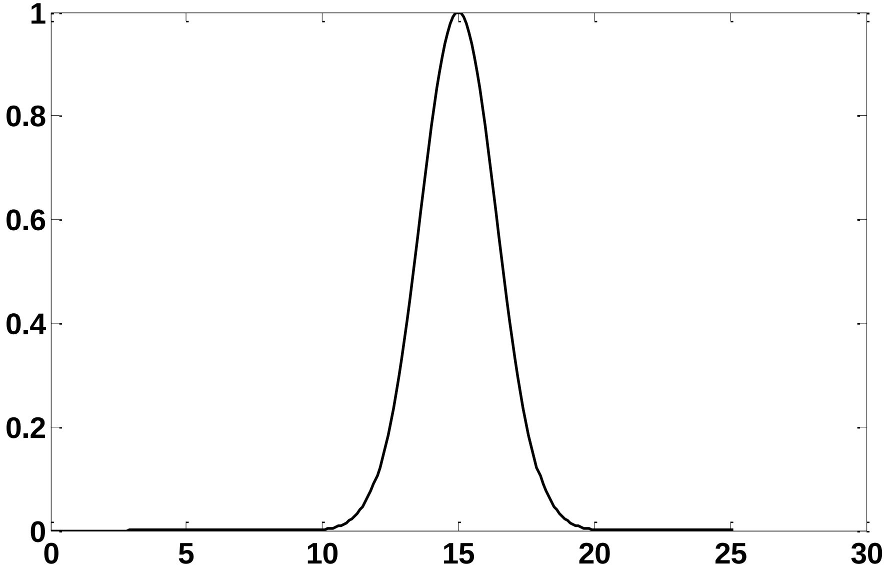
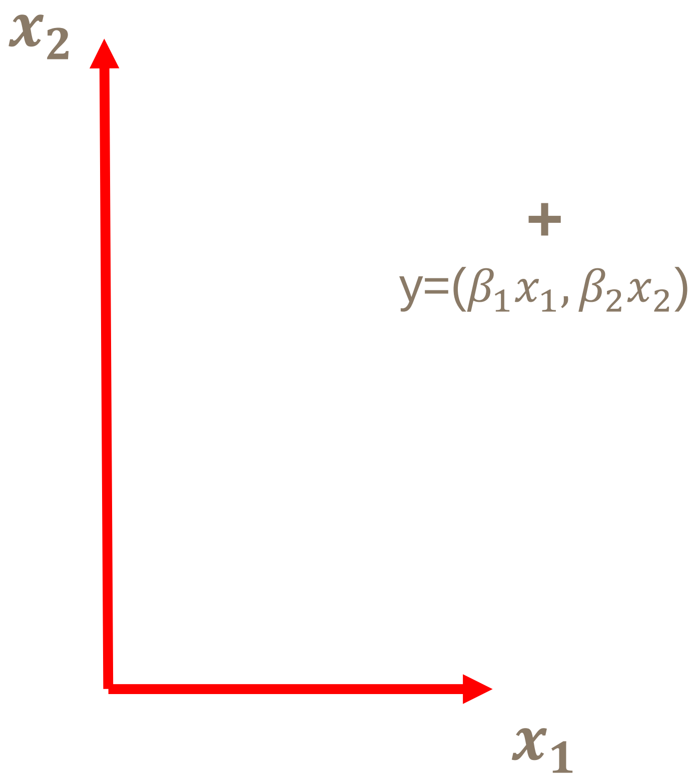




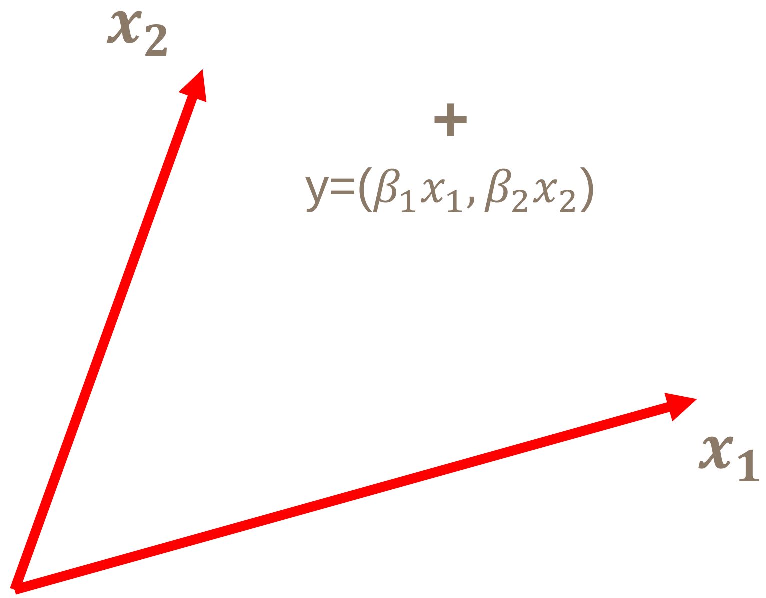




Geometric interpretation of basis functions



What if basis functions are not orthogonal?



Linearly dependent equations

$$y_1 = x_{11} * \beta_1 + x_{12} * \beta_2$$
$$y_2 = x_{21} * \beta_1 + x_{22} * \beta_2$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \mathbf{X} * \boldsymbol{\beta}$$

can be interpreted as

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} * \beta_1 + \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} * \beta_2$$

Solving Linear Equations

Problem:

- We have an equation **Mx**=**y**
- We know M and y
- We want to know x

We need a matrix $\mathbf{M}^{-1}$ with the property

$$\mathbf{M}^{-1}*\mathbf{M} = \mathbf{I}$$

(**I** is the identity matrix)

because then:

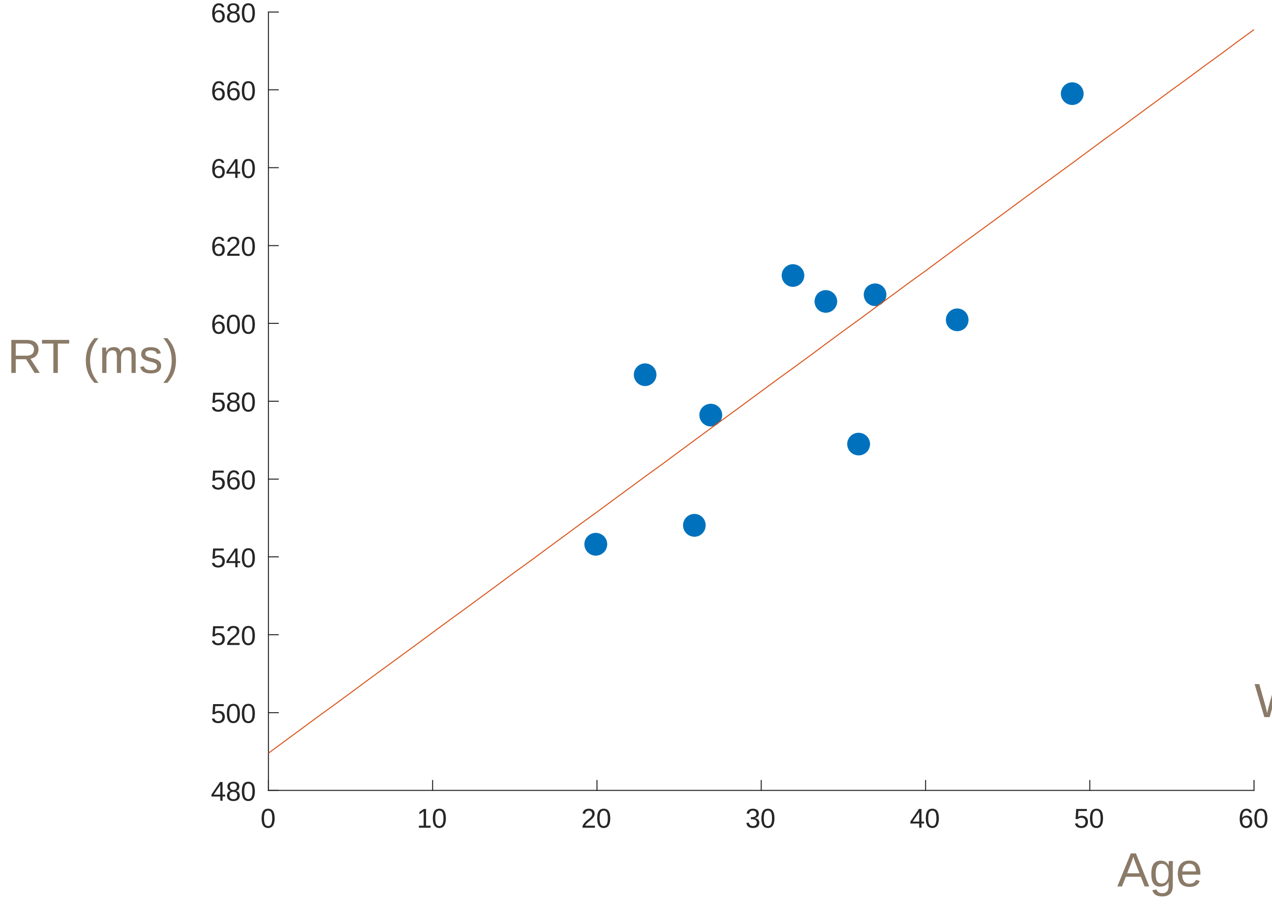
$$\mathbf{M}^{-1}*\mathbf{M}\mathbf{x} = \mathbf{I}*\mathbf{x} = \mathbf{x} = \mathbf{M}^{-1}\mathbf{y}$$

$\mathbf{M}^{-1}$ is the “inverse matrix” **of M**

M only has an inverse matrix (is “invertible”) when there are no pairs of columns and pairs of rows that are perfectly correlated (i.e. they are “linearly independent”).

Linear Regression – fewer unknowns than data points

(“overdetermined problem”)



One variable, multiple data points:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \cdots \\ y_{10} \end{pmatrix} = \begin{pmatrix} x_1 \\ \cdots \\ x_{10} \end{pmatrix} * \beta \; = \mathbf{x} * \beta$$

Cheating alert:
We assume the intercept has been subtracted from the data.

Linear Regression – fewer unknowns than data points

The β that minimises the least-squares error in this equation can be computed using the "pseudoinverse":

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \dots \\ y_{10} \end{pmatrix} = \begin{pmatrix} x_1 \\ \dots \\ x_{10} \end{pmatrix} * \beta \;\; = \mathbf{x} * \beta$$

$$\beta = \mathrm{pinv}(\mathbf{x}) * \mathbf{y}$$

Multiple Linear Regression

2 parameters, 10 data points:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \dots \\ y_{10} \end{pmatrix} = \begin{pmatrix} X_{1,1} \\ \dots \\ X_{10,1} \end{pmatrix} * \beta_1 + \begin{pmatrix} X_{1,2} \\ \dots \\ X_{10,2} \end{pmatrix} * \beta_2 = \begin{pmatrix} X_{1,1} & X_{1,2} \\ \dots & \dots \\ X_{10,1} & X_{10,2} \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \mathbf{X} * \boldsymbol{\beta}$$

where the design matrix **X** has dimension (10,2) and the parameter vector beta has dimension (2).

$$\boldsymbol{\beta} = \mathbf{pinv}(\mathbf{X}) * \mathbf{y}$$

where pinv(**X**) has dimension (2,10).

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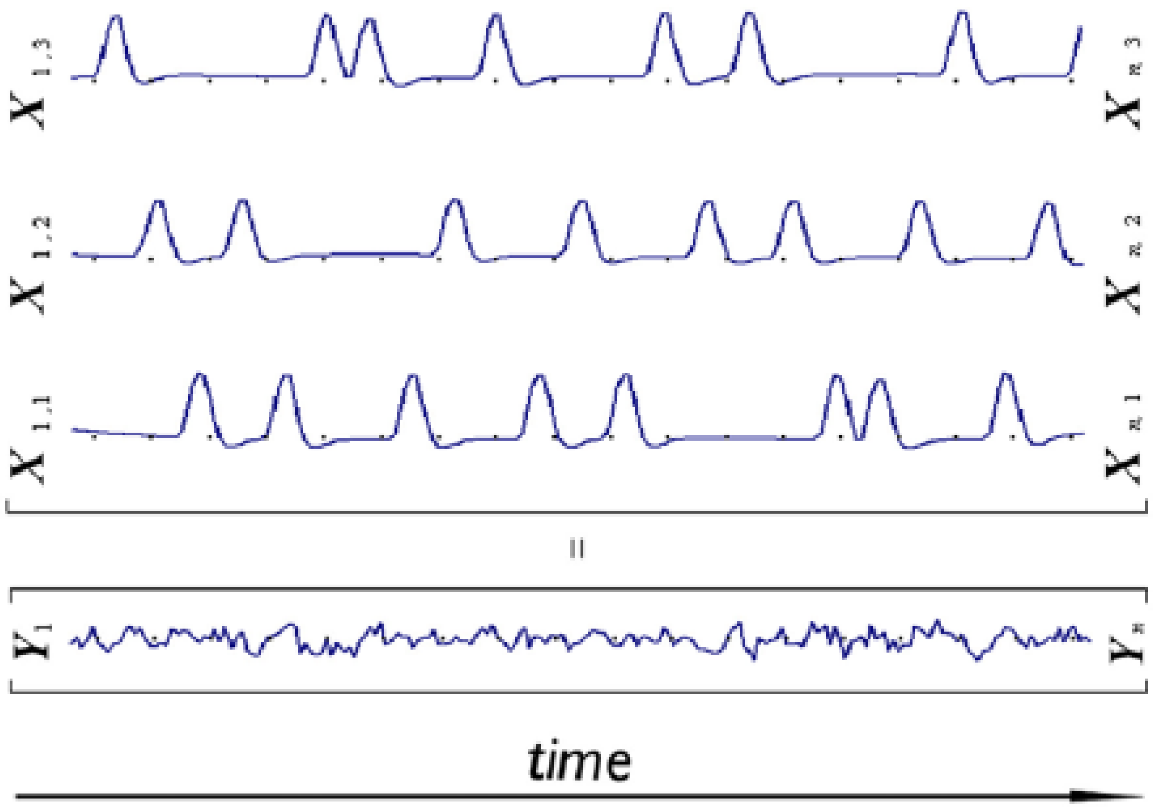
for event type 1

Predicted time course
for event type 2

Predicted time course
for event type 3

BOLD time course
in one voxel



measured time series

parameter estimates

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

design matrix

More unknowns than data points
("underdetermined problem")

$$y_1 = x_1{*}\beta_1 + x_2 * \beta_2$$

e.g.:

$$1 = 1{*}\beta_1 + 1 * \beta_2$$

i.e.

$$1 = \begin{pmatrix} 1 & 1 \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

More unknowns than data points
("underdetermined problem")

$$1 = \begin{pmatrix} 1 & 1 \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

The unique solution that minimises the "L2-norm", i.e.

$$\beta_1^2 + \beta_2^2 \rightarrow minimal$$

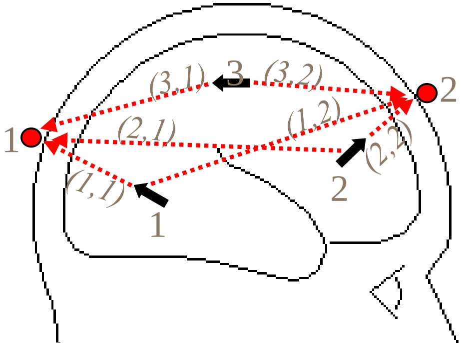
is

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

"Minimum-norm solution"

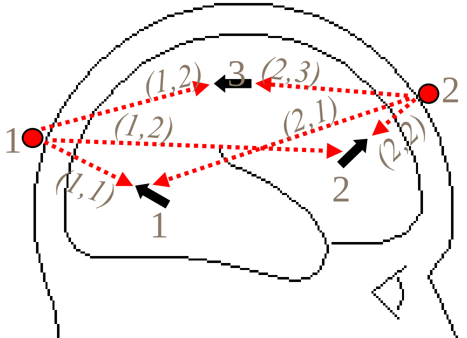
The EEG/MEG Inverse Problem Is Underdetermined

Forward Problem



$$\mathbf{d} = \mathbf{L}\mathbf{j}$$

Linear Inverse Problem



$$\hat{\mathbf{j}} = \mathbf{G}\mathbf{d}$$

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More unknowns than data points
(“underdetermined problem”)

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The β that minimises the sum of least-squares for $\boldsymbol{\beta}$ in this equation can be computed using the pseudoinverse (“minimum-norm solution”):

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \\ X_{3,1} & X_{3,2} \end{pmatrix} * \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \mathbf{X} * \boldsymbol{\beta}$$

$$\beta = \mathrm{pinv}(\mathbf{X}) * \mathbf{y}$$