MRC | Cognition and Brain Sciences Unit

Introduction to Matrix Algebra:

Matrices, vectors, and what you can do with them.

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Why Matrix Algebra?

Matrix notation originally invented to express linear algebra relations (*Cayley & Sylvester, Cambridge 1858*)

$$\begin{aligned} a_{11}x_1 + \quad a_{12}x_2 + \quad a_{13}x_3 &= y_1 \\ a_{21}x_1 + \quad a_{22}x_2 + \quad a_{13}x_3 &= y_2 \end{aligned}$$

- Compact notation for describing sets of data & sets of linear equations.
- Enhances visualisation and understanding of essentials.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Efficient for manipulating sets of data & solving sets of linear equations.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}; \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix};$$

$$\mathbf{Ax = y}$$

- Translates directly to the implementation of linear algebra processes in languages that offer array data structures (e.g. **MATLAB**).

```
A = [2 4;1 7]; x = [3 ;2];
Y = A*x;
```

Basics: Taxonomy

Matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ **Vector** $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Matrix : A collection of numbers ordered by rows and columns.
Example: a 2 rows by 3 columns matrix.

Square matrix

$$\begin{pmatrix} 9 & 1 & 1 \\ 1 & 3 & 7 \\ 5 & 7 & 2 \end{pmatrix}$$

```
A = [9 1 1; ...
 1 3 7; 5 7 2];
```

Symmetric matrix

$$\begin{pmatrix} 9 & 1 & 5 \\ 1 & 3 & 7 \\ 5 & 7 & 2 \end{pmatrix}$$

```
S = A;
S(1,3)=5;
```

Identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
I = eye(3,3);
```

Diagonal matrix

$$\begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

```
D =I .* A;
```

Zero matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
Z = zeros(3,3);
```

All-ones matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

ones(3,3);
OR Z+1;

Vector: In most cases a vector can be defined as a one-dimensional matrix (Matlab always does!).

Column Vector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

```
C = [1 ; 2];
```

Row Vector

$$(x_1 \; x_2)$$

```
V = [1 2];
```

Basics

The dimension (order) of a matrix is given by the number of its rows and columns.
Example: 2 rows x 3colums

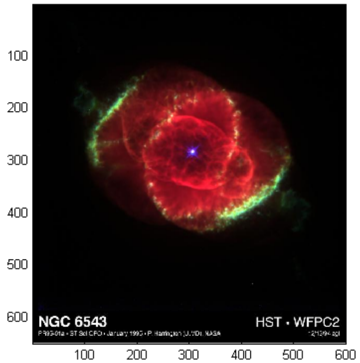
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

```
Order = size(A)
Nrows = size(A,1)
Ncols = size(A,2)
```

NOTE: Matlab uses **multidimensional arrays** which are an extension of the normal 2-dimensional matrix

$$\begin{pmatrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \end{pmatrix} \begin{pmatrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \end{pmatrix} \cdots\cdots \begin{pmatrix} a_{11n} & a_{12n} & a_{13n} \\ a_{21n} & a_{22n} & a_{23n} \end{pmatrix}$$

Example: *colour images in Matlab are 3-D arrays. The 3rd dimension encodes the primary colours (i.e. Red, Green, Blue).*




```
RGB = imread('ngc6543a.jpg');
image(RGB); axis image;

size(RGB)
```

Try it out

```
GB = RGB; GB(:,:,1)=0;
RB =  RGB; RB(:,:,2)=0;
Etc…
image(….); axis image;
```

Operations

Transposition:

$$a_{ij} \rightarrow a_{ji}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 8 \\ 0 & 4 & 2 \end{pmatrix}; \; \mathbf{A}^T = \begin{pmatrix} 1 & 0 \\ 3 & 4 \\ 8 & 2 \end{pmatrix}$$

```
At = A';
```

Addition/Subtraction:

Matrices/vectors need to have the ***same dimensions*** *(i.e. nrows & ncols).*

$$a_{ij} \pm b_{ij} = r_{ij}$$

$$\overset{A}{\begin{pmatrix} 1 & 3 & 8 \\ 0 & 4 & 2 \end{pmatrix}} + \overset{B}{\begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix}} = \overset{R}{\begin{pmatrix} 2 & 4 & 10 \\ 3 & 9 & 4 \end{pmatrix}}$$

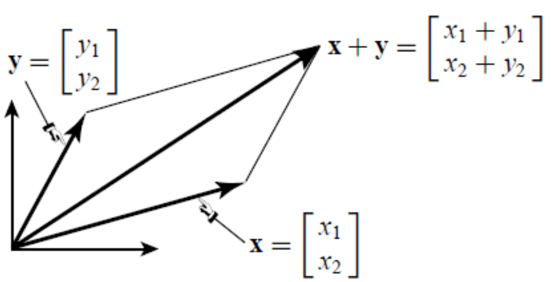
```
RGB + RGB(:,:,1);%WRONG
RGB + RGB(:,:,:);%RIGHT
```

Properties of ***addition****:*

- ***Commutative:*** *A+B = B+A*
- ***Associative:*** *A+(B+C) = (A+B)+C;*

Geometric interpretation (parallelogram law)

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \; \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \; \mathbf{x+y}$$




```
x = [5;2]; y = [2;5]
plot([0,x(1)],[0 ,x(2)],'b');hold on;
plot([0,y,(1)],[0,y(2)],'r');

xy = x+y;
plot([0,xy(1)],[0,xy(2)],'- - k');
```

Operations: Multiplication (& Division)

Multiplication by scalar:

$$c * \boldsymbol{A} = c * a_{11} \dots c * a_{nn}$$

$$3*\mathbf{A} = 3*\begin{pmatrix} 1 & 3 & 8 \\ 0 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 24 \\ 0 & 12 & 6 \end{pmatrix}$$

*NOTE: **Division** is equivalent to multiplication by 1/c (e.g. 1/3).*

Geometric interpretation

*This operation is also called **scaling of a vector**: the scaled vector points the same way, but its magnitude is multiplied by c.*

```
V = [2 5];
C = 2;  sV = C*V;
plot([0 V(1)],[0 V(2)],'r');
hold on;
plot([0 sV(1)],[0 sV(2)],'b');
```

*If **c < 0** the direction of the vector is reversed (reflexion about the origin).*

Operations: Multiplication (& Division)

Inner product (or scalar product) of two column vectors (of same order)

$$X^T Y = Y^T X = \sum_{i=1}^{n} x_i y_i$$

$X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $Y = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$; $X^T Y = (2\ 3) \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 2x1 + 3x5 =$ **17**

Properties: ***Commutative***

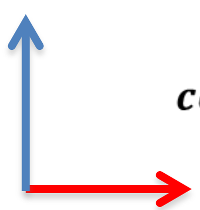
NOTE: *Matlab's " **.*** " is an **Array operator** that multiplies two vectors of the same order element by element. **XY** = **Z** -> size(**Z**) = size(**X**) = size(**Y**);*

```
X = [2; 5]; Y = V;
V'*Y
V.*Y
```

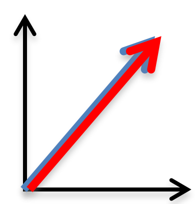
Geometric interpretation

The angle in radians between two arbitrary vectors is defined as

$$cos\theta = \frac{(\boldsymbol{x}, \boldsymbol{y})}{|\boldsymbol{x}||\boldsymbol{y}|}$$



cosθ = 0

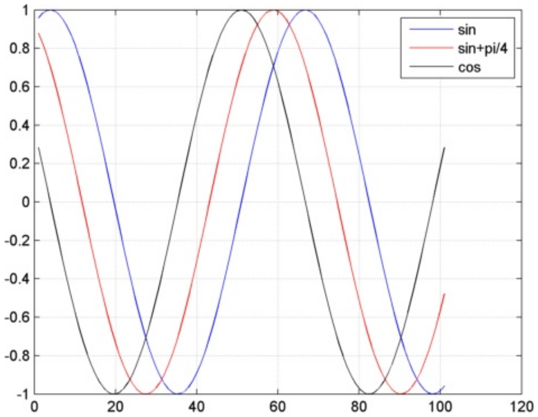


cosθ = 1

*The cosine function is closely related to **covariance***

Example

```
% generate 3 sinusoids of different phases
Phi = [0, pi/4, pi/2];
X =  [-5:0.1:5]’; %NOTE: transposition
S1 = sin(x+Phi(1)); S2 = sin(x+Phi(2)); S3 = sin(x+Phi(3)); % NOTE: we should have 3 column vectors, check!
%Plot them
plot(S1,’b’);hold on; plot(S2,’r’);plot(S3,’k’);
```




```
% Create an anonymous function to calculate the Euclidean norm
Enorm = @(x) sqrt(sum(x.^2))

%Calculate the cosine between vectors
C1 = (S1’*S2)/(Enorm(S1)*Enorm(S2));
C2 = ….
C3 = ….
```

$$cos\theta = \frac{(\boldsymbol{x}, \boldsymbol{y})}{|\boldsymbol{x}||\boldsymbol{y}|}$$

Operations: Multiplication

Multiplication of matrix with vector:

$$y = Ax;\ y_i = \sum_{j=1}^{n} a_{ij}\, x_j \quad i = 1..m$$

3 Columns, 3 Rows

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix} * \begin{pmatrix}3\\ 4\\ 5\end{pmatrix} = \begin{pmatrix}1*3 + & 1*4 + & 1*5\\ 2*3 + & 2*4 + & 2*5\end{pmatrix} = \begin{pmatrix}12\\ 24\end{pmatrix}$$

Remember that earlier we multiplied row vectors with column vectors?
This makes sense now, because vectors are special cases of matrices.

Multiplication of matrix with matrix:

$$C = AB;\ c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} \quad i = 1..m, k = 1,..p.$$

Every element ik of C is the scalar product of the i-th row of A with the k-th column of B

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix} * \begin{pmatrix}3 & 3 & 5\\ 4 & 4 & 5\\ 5 & 4 & 5\end{pmatrix} = \begin{pmatrix}12 & 11 & 15\\ 24 & 22 & 30\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix} * \begin{pmatrix}3\\ 4\\ 5\end{pmatrix} = \begin{pmatrix}12\\ 24\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix} * \begin{pmatrix}3\\ 4\\ 4\end{pmatrix} = \begin{pmatrix}11\\ 22\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{pmatrix} * \begin{pmatrix}5\\ 5\\ 5\end{pmatrix} = \begin{pmatrix}15\\ 30\end{pmatrix}$$

Properties:

Associative: A(BC) = (AB) C

Distributive: A(B+C) = AB + AC

Operations:

Inverse of a (square) matrix

In scalar algebra, the inverse of a number x is x^{-1} so that $x*x^{-1} = 1$.
In matrix algebra the inverse of a matrix is that matrix that multiplied by the original matrix gives an identity matrix: $\boldsymbol{AA^{-1} = A^{-1}A = I}$

*A matrix must be square, but not all square matrices have an inverse (e.g. **singular matrices**).*

```
IA = inv(A)
A*IA
```

Example: simple linear regression

Regression Coefficient

$$y = \beta_0 + \beta_1 x + \varepsilon$$

Intercept **Error term**

→

$$\overset{Y}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \overset{X}{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}} \overset{B}{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}$$

→ **Y = XB**

```
load accidents
x = hwydata(:,14); %Population of states
X = [ones(length(x),1) x];%add a column of 1s to calculate intercept
Y = hwydata(:,4); %Accidents per state
format long
```

" \ " operator mldivide : solve systems of linear equations Y=BX for B (similar to $X^{-1}Y$)

```
B = X\Y
yCalc = X*b; %NOTE: vector by matrix product

scatter(x,y)
hold on;
plot(x,yCalc,'-')
legend('Data','Fitted function','Location','best');
xlabel('Population of state')
ylabel('Fatal traffic accidents per state')
```

… the end, thanks!