Filtering and Oscillations

Olaf Hauk

MRC Cognition and Brain Sciences Unit
olaf.hauk@mrc-cbu.cam.ac.uk

Periodic Signals

A periodic signal repeats itself with a period T:

$$f(t+T) = f(t)$$

This is the case, for example, for sine and cosine functions:

$$\text{In radians:}$$
$$\sin(t+2\pi) = \sin(t)$$
$$\cos(t+2\pi) = \cos(t)$$

$$\text{In degrees:}$$
$$\sin(t+360) = \sin(t)$$
$$\cos(t+360) = \cos(t)$$

$$\text{Matlab does radians.}$$
$$360^{o} \text{ corresponds to } 2\pi\text{, therefore:}$$
$$\text{to convert x in radians to degrees:} (360/2\pi) * x = (180/\pi) * x$$
$$\text{to convert x in degrees to radians:} (2\pi/360) * x = (\pi/180) * x$$

Sine, Cosine and Cousins

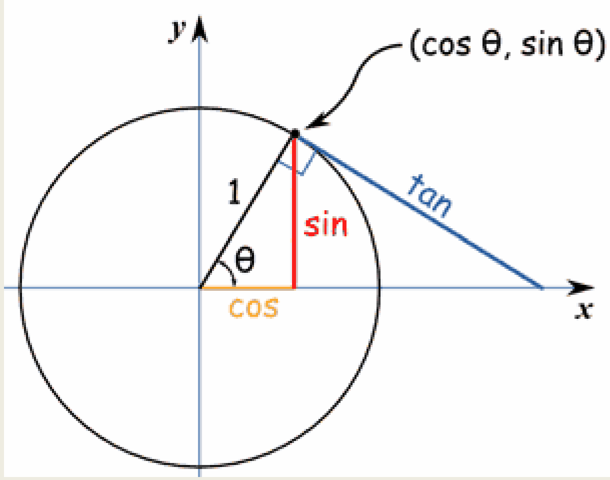
Important for oscillations, filters, Fourier Transform etc.

$$f(x) = a * \sin(b * x + c)$$

$$f(x) = a * \cos(b * x + c)$$

$$a : \text{amplitude}, b : \text{frequency}, \text{c} : \text{phase}$$

$$\text{Then there is the tangens}: \ f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$



θ takes the role of x, the lengths of the coloured lines are the function outputs

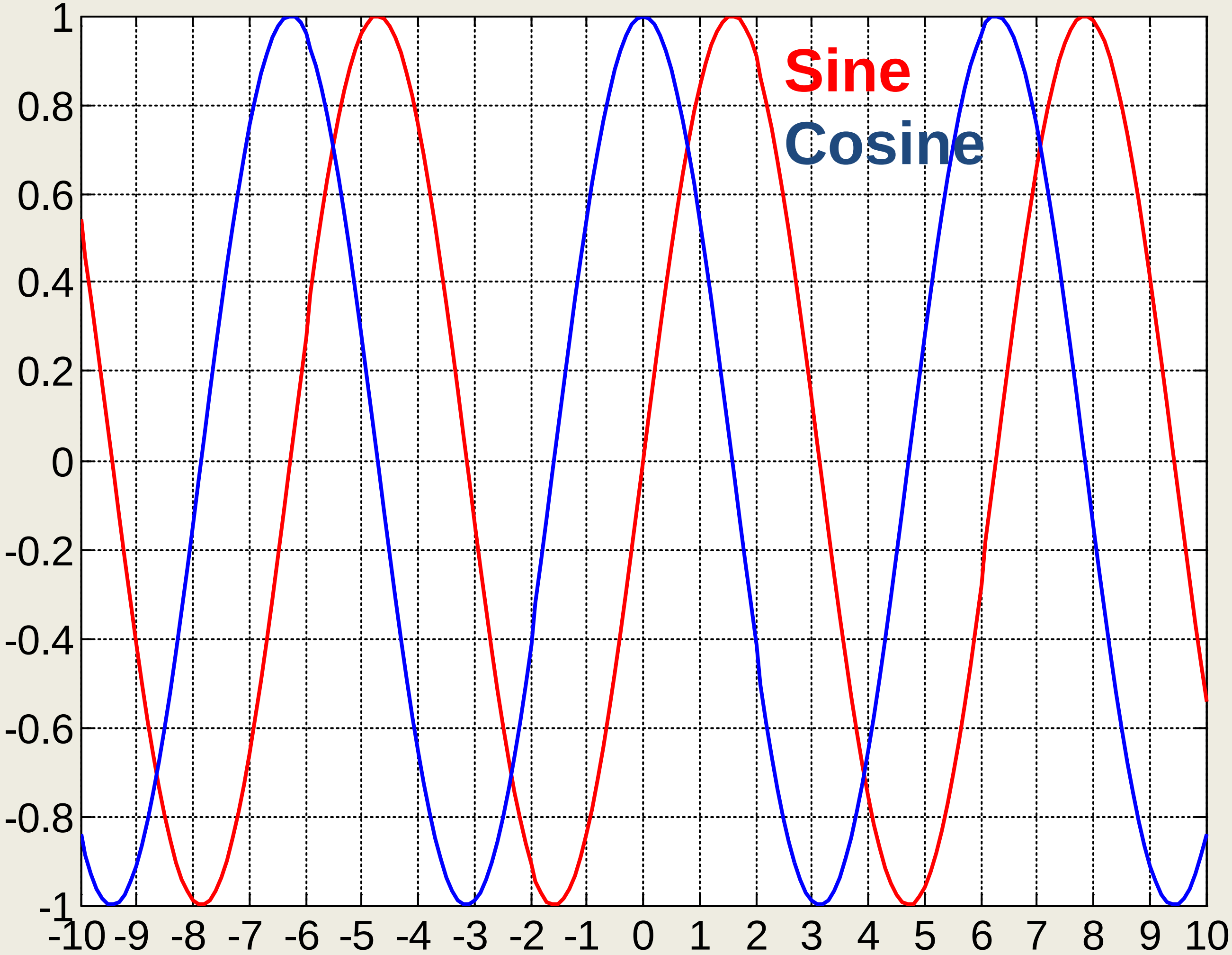
Inverse of sine and cosine: arcsine and arccosine

https://www.youtube.com/watch?v=z82I6u4DFTo

Examples

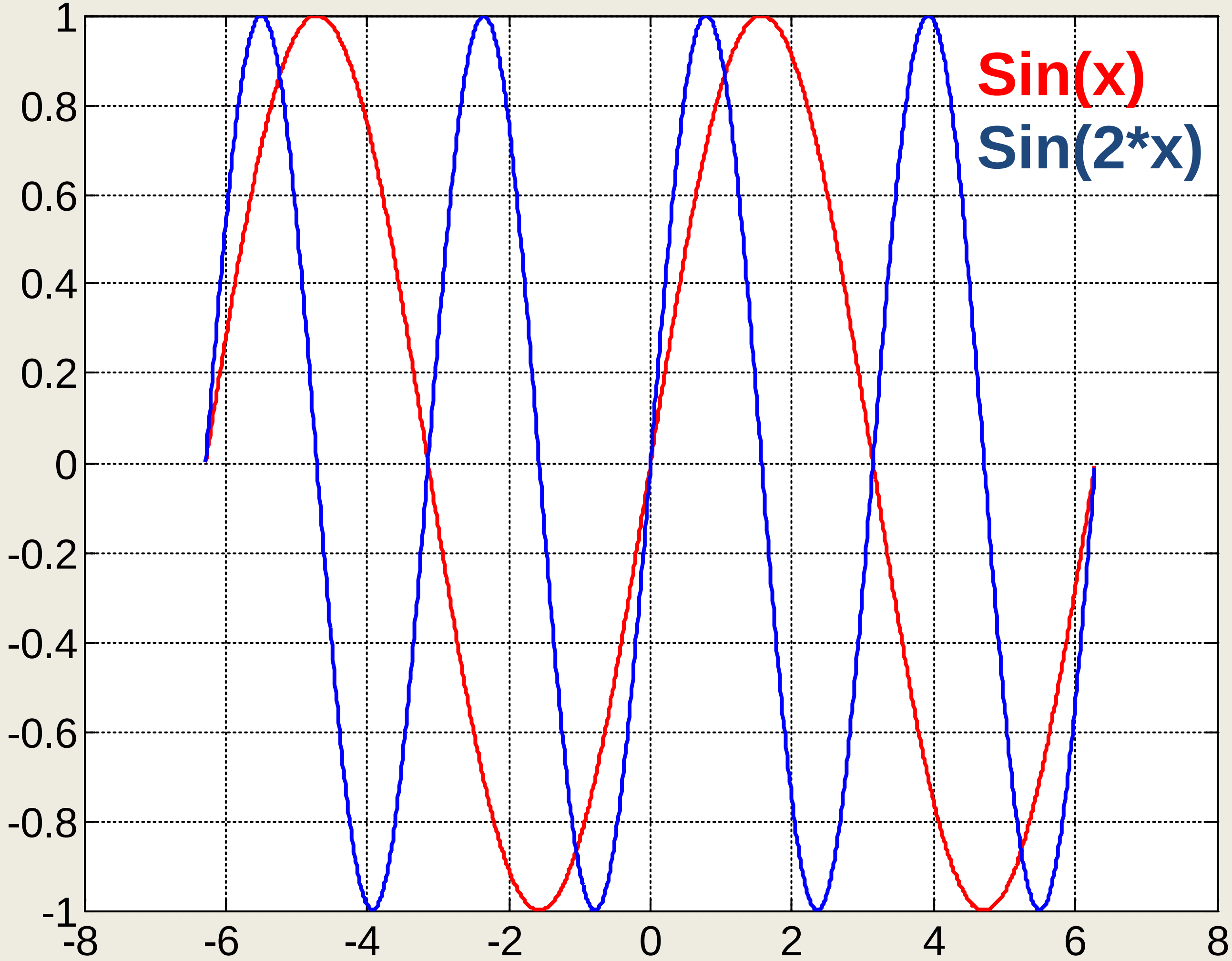
Sine and Cosine are Orthogonal

(for same phase)



$$\int_{-\infty}^{\infty} \sin(x)\cos(x)dx = 0$$

Sine/Cosine At Integer Frequency Intervals Are Orthogonal



$$\int_{-\infty}^{\infty} \sin(f * x)\sin(n * f * x)dx = 0$$

(Fast) Fourier Transform in Words

What you want:

You've got a signal consisting of N sample points (equidistant).
You want to know which frequencies contribute to the signal, and how much.

In other words:
You want to describe your signal as a linear combination of sines and cosines, ideally of orthogonal basis functions made up of sines and cosines.

What you've got:

With N samples, you can estimate at most N independent parameters.

You cannot estimate frequencies above half of the sampling frequency SF (Nyquist).

For a given frequency, sine and cosine are orthogonal,
i.e. 2 basis functions per frequency.

Basic Idea of (Discrete) Fourier Transform:

Divide the frequency range 0 to SF/2 evenly into N/2 frequencies.

For every frequency, create a sine and a cosine.

Use these (orthogonal) sines and cosines as your basis functions.

Project these basis functions onto your data, get the amplitudes for individual basis functions.

– done!

Fast Fourier Transform (FFT): A fast algorithm to do this.

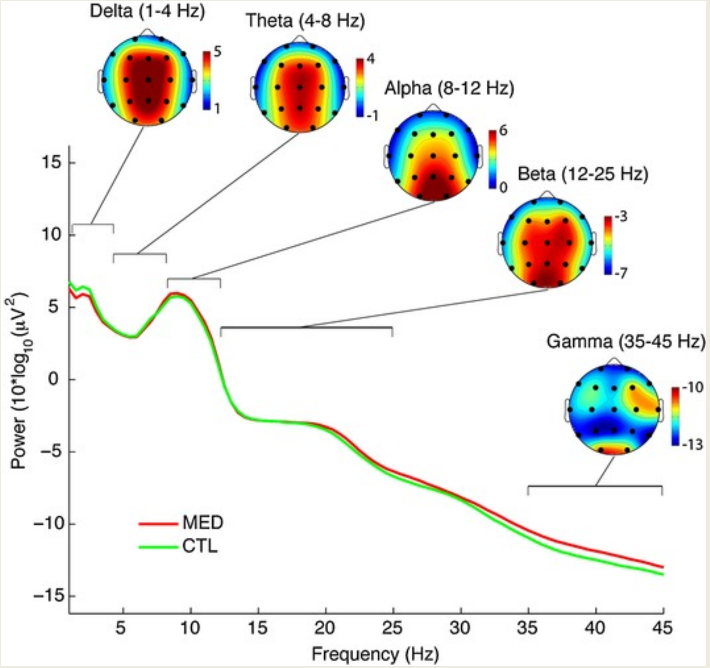
(I’m cheating a bit, assuming an appropriate N and ignoring the mean. But the principle is ok.)

Power Spectral Density

Plot the “power per frequency bin”, or the power spectral density (power per Hz).

Often we don’t plot power, but something related/proportional to it, e.g. V^2 or fT^2.
If we take the square root to get back to the original measurement units (e.g. V or fT), then we get for example $V/\sqrt{Hz}$.

dB



http://link.springer.com/article/10.1007%2Fs10339-009-0352-1/

Filtering

Basic idea:
Transform your data into frequency domain
(e.g. get the coefficients for sine and cosine basis functions above, FFT)

Remove what you don't want
(e.g. set coefficients of unwanted basis function to zero)

Reconstitute your data with the new weightings of basis functions
(e.g. using inverse FFT)

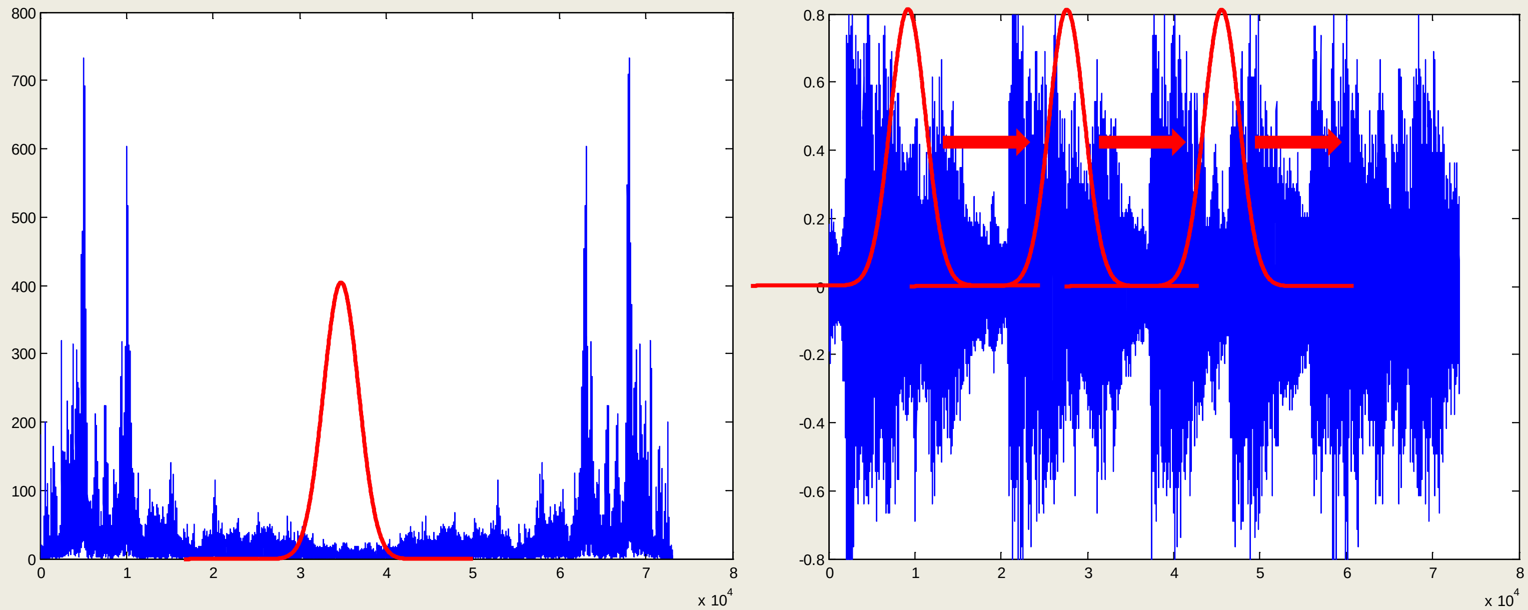
All this is usually more complicated in practice, because the above is based on some idealising assumptions.

Filtering

Every operation in the frequency domain has a pendant in the time domain:

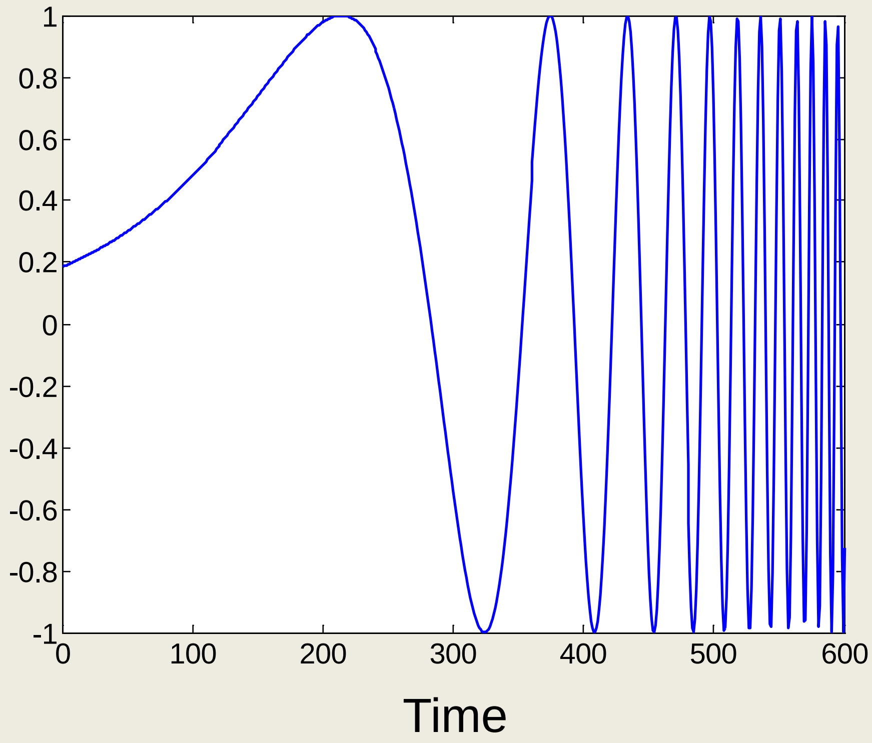
Multiplication in the frequency domain is convolution in the time domain,
and vice versa (convolution theorem).

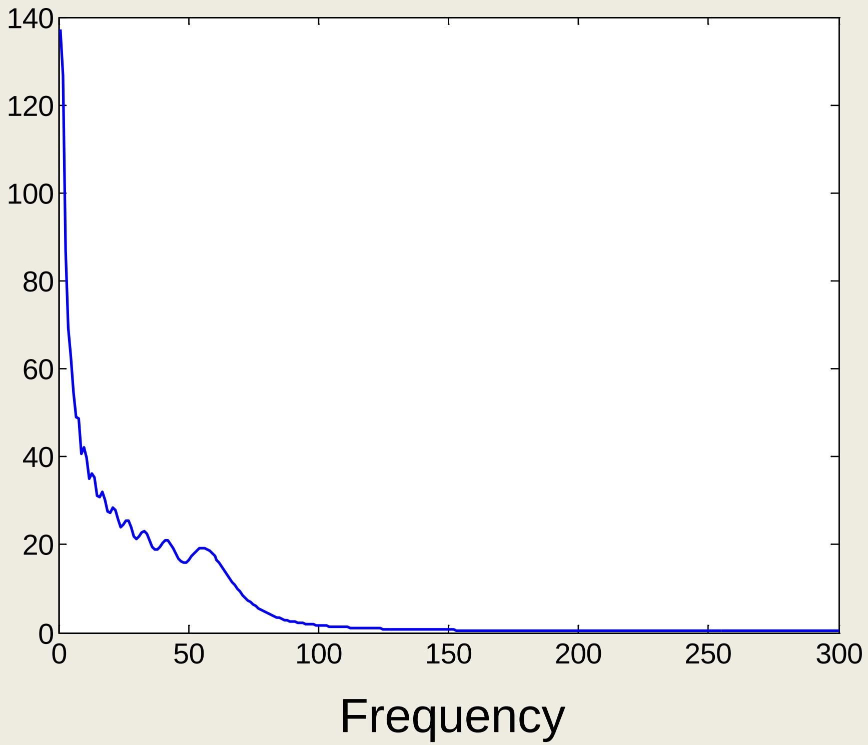
For example, band-pass filtering in the frequency domain using a Gaussian filter
is equivalent to convoluting (smoothing) in the time domain using a Gaussian kernel



Time-Frequency Analysis

Fourier Transform assumes constant sines and cosines across the whole time series.
But what does an FFT mean for a signal like this?

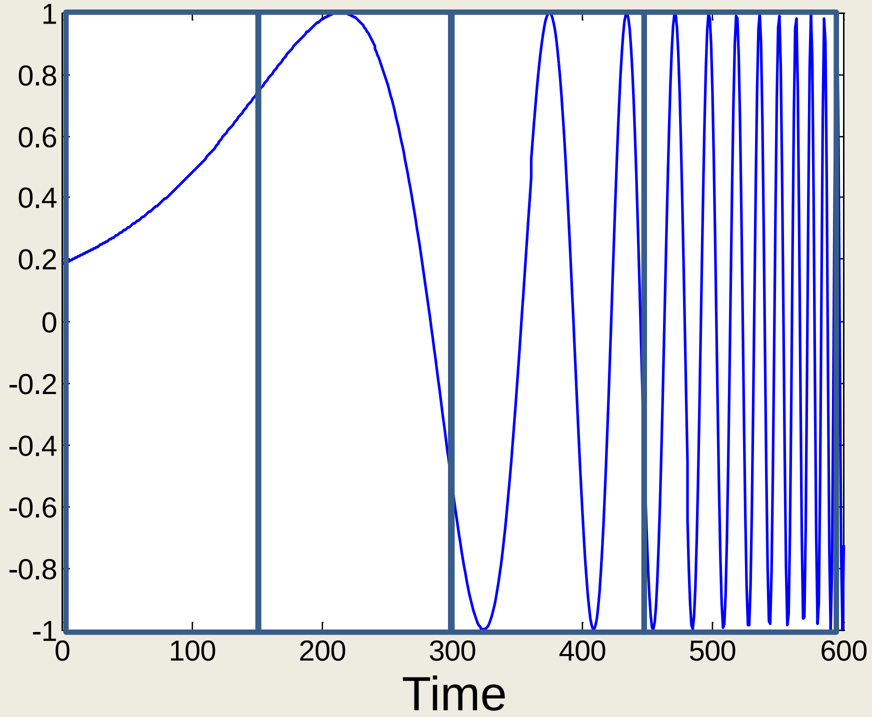




Time-Frequency Analysis

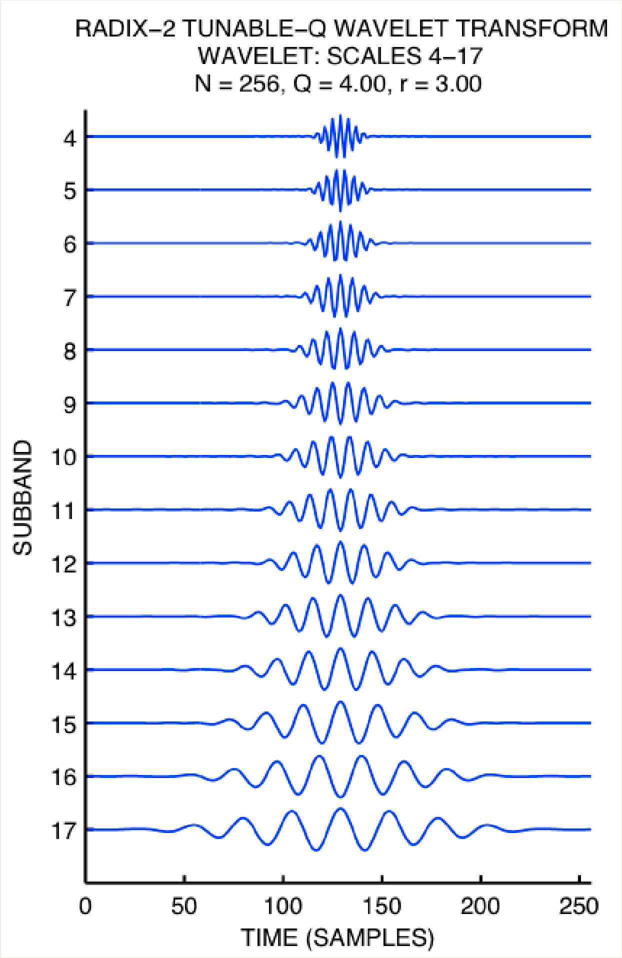
You could run separate FFTs for different (sliding) time windows:

But different window sizes are more or less optimal for different frequencies.
Run different FFTs with different window sizes for different frequency ranges? Ouff.

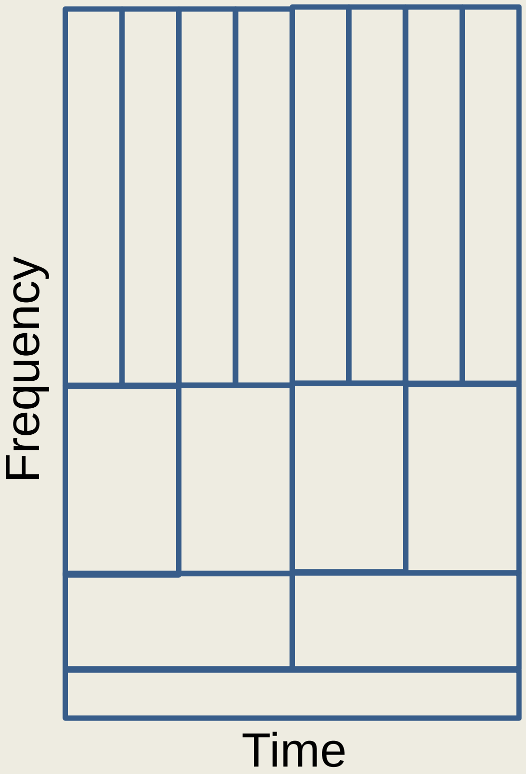


Time-Frequency Analysis: Wavelets

Wavelets provide an optimal trade-off between frequency and time resolution.



Wavelets are convolved with the data to give instantaneous amplitude and phase estimates for different frequency ranges.



Frequency
Time